AN APPROXIMATION FORMULA FOR HOLOMORPHIC FUNCTIONS BY INTERPOLATION ON THE BALL

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Abstract. We deal with a problem of the reconstruction of any holomorphic function f on the unit ball of $\mathbb{C}^2$ from its restricions on a union of complex lines. We give an explicit formula of Lagrange interpolation's type that is constructed from the knowledge of f and its derivatives on these lines. We prove that this formula approximates any function when the number of lines increases. The motivation of this problem comes also from possible applications in mathematical economics and medical imaging.

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1. Introduction

In this paper, we deal with the following problem of reconstruction: f being a holomorphic function on a bounded domain $\Omega \subset \mathbb{C}^m$, we want to reconstruct f from its restriction on an analytic subvariety Z of Ω. We assume that this analytic subvariety is given by $Z = \{z \in \Omega,\ g(z) = 0\}$, where $g \in \mathcal{O}\left(\overline{\Omega}\right)$.

A natural way is to construct from the restriction $f_{\{g=0\}}$ a holomorphic function $\widetilde{f} \in \mathcal{O}\left(\overline{\Omega}\right)$ that interpolates f, i.e. $\widetilde{f}(z) = f(z)$, $\forall\, z \in Z$ (see [3], [2], [5]). Nevertheless, we know that generally $\widetilde{f} \neq f$ then we cannot regain f. This yields to the following questions: what will happen if we make larger the analytic set Z where f and $\widetilde{f}$ coincide ? Will $\widetilde{f}$ converge to f ? Else is it only true for a certain class of functions f ? Is there also an explicit formula that gives $\widetilde{f}$ or another that approximates f ? And what could be the precision of this approximation ?

A first consequence is that $f - \widetilde{f}_N$ will vanish on the set Z_N, $N \geq 1$, that is increasingly big but this is not sufficient to deduce that $f - \widetilde{f}_N \to 0$. Nevertheless, it is sufficient to prove that $\widetilde{f}_N$ is uniformly bounded on any compact subset of Ω.

Indeed, by the Stieltjes-Vitali-Montel theorem, one can choose a subsequence (precisely, choose a subsequence from any subsequence) that converges to a holomorphic h that vanishes on $\bigcup_{N\geq 1} Z_N$, then $h=0$ (for all subsequence) and $\widetilde{f}_N \to f$.

We begin with a special case in the unit disc $D(0,1)\subset\mathbb{C}$. The analytic set Z is given by a sequence $\{\eta_j,\ j\geq 1\}\subset D(0,1)$ and the data of $f_Z,\ f\in\mathcal{O}\left(\overline{D}(0,1)\right)$, is the sequence $\{f(\eta_j),\ j\geq 1\}$. Now consider, for all $\eta\in D(0,1)$ the Blaschke function $\varphi_\eta\in\mathcal{O}\left(\overline{D}(0,1)\right)$ defined as

$$\varphi_\eta(z) \quad := \quad \frac{z-\eta}{1-\overline{\eta}z}\,.$$

One has, for all $N\geq 1$ and all $z\in D(0,1)$,

$$\prod_{l=1}^N \varphi_{\eta_l}(z)\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)\,d\zeta}{\prod_{l=1}^N\varphi_{\eta_l}(\zeta)(\zeta-z)} \quad = \quad f(z)-\sum_{l=1}^N\prod_{j\neq l}\frac{\varphi_{\eta_j}(z)}{\varphi_{\eta_j}(\eta_l)}\,\frac{f(\eta_l)}{1-\overline{\eta_l}z}\,.$$

Since $|\varphi_\eta(\zeta)|=1$ for all $|\zeta|=1$ and $|\varphi_\eta(z)|<1$ on $D(0,1)$, one has, for all compact subset $K\subset D(0,1)$,

$$\left|\prod_{l=1}^N \varphi_{\eta_l}(z)\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)\,d\zeta}{\prod_{l=1}^N\varphi_{\eta_l}(\zeta)(\zeta-z)}\right| \;\leq\; \frac{\sup_{\zeta\in K}|f(\zeta)|}{\varepsilon_K}(1-\varepsilon_K)^N \xrightarrow[N\to\infty]{} 0\,.$$

It follows that f can be approximated by the above explicit interpolation formula. Notice that it is of Lagrange interpolation's type (see below). In addition, we know the precision of this approximation.

This example gives an idea for the principal result of this paper. Nevertheless, in order to generalize this method to $\Omega\subset\mathbb{C}^m$, we need to find functions φ_Z of Blaschke type such that $Z=\{\varphi_Z=0\}$, i.e. that satisfie as well $|\varphi_Z(\zeta)|=1,\ \forall\,\zeta\in\partial\Omega$. This will be not possible in our case since we will consider subvarieties that cross $\partial\Omega$.

Therefore we begin with a preliminar result (section 2, proposition 1) that gives the essential idea. Let be $f\in\mathcal{O}\left(\overline{\Omega}\right)$ and assume that $Z=\{g(z)=0\}$ where $g\in\mathcal{O}\left(\overline{\Omega}\right)$. Then for all $z\in\Omega$,

$$f(z) \quad = \quad Res(f,g)(z)+PV(f,g)(z)\,, \tag{1.1}$$

where $Res(f,g)$ is a holomorphic function that interpolates f on $\{g=0\}$ and is constructed with the residual current of $1/g$ (resp. $PV(f,g)$ is constructed with the principal value current of $1/g$, see [4], [7]).

In all the following, we will deal with the unit ball of $\mathbb{C}^2$,

$$\mathbb{B}_2 \quad = \quad \left\{(z_1,z_2)\in\mathbb{C}^2,\ |z_1|^2+|z_2|^2<1\right\}\,.$$

We also consider for $Z=\{g(z)=0\}$ a union of lines that cross the origin. Without loss of generality, one can choose

$$g_n(z) \quad = \quad z_1^{m_1}\prod_{j=2}^{n-1}(z_1-\eta_j z_2)^{m_j}\,z_2^{m_n}\,, \tag{1.2}$$

where $n\geq 3,\ m_1,\dots,m_n\in\mathbb{N}$ and $0<|\eta_2|\leq\cdots\leq|\eta_{n-1}|$ (one has $\eta_1=0$ and by convention $\eta_n=\infty$).

On the other hand, we specify that, for all $f \in \mathcal{O}(\mathbb{B}_2)$ and all $m_p \geq 2$, the data $f_{\{g_n(z)=0\}}$ is defined as

$$f_{\{z_1^{m_1}=0\}},\ \left(f_{\{(z_1-\eta_p z_2)^{m_p}=0\}}\right)_{2\leq p\leq n-1},\ f_{\{z_2^{m_n}=0\}} \tag{1.3}$$

where $f_{\{(z_1-\eta_p z_2)^{m_p}=0\}}$ is defined as

$$f_{\{z_1=\eta_p z_2\}}, \left\{\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}\right\}_{\{z_1=\eta_p z_2\}}, \dots, \left\{\left(\frac{\partial^{m_p-1} f}{\partial z_1^{j_1}\partial z_2^{j_2}}\right)_{j_1+j_2=m_p-1}\right\}_{\{z_1=\eta_p z_2\}} \tag{1.4}$$

(as well as for $f_{\{z_1^{m_1}=0\}}$ and $f_{\{z_2^{m_n}=0\}}$).

The problem of interpolation by lines is motivated by applications in mathematical economics and medical imaging where we have to reconstruct any function F with compact support from knowledge of its Radon transform $(RF)(\theta^{(p)}, s)$, $(\theta^{(p)}, s) \in S^{n-1}\times\mathbb{R}$, on a finite number of directions $\theta^{(p)}$, $p = 1\dots,n$ (see [6]).

Before giving the principal result of this paper, we need to specify the following notations.

First, $W \subset \mathbb{C}^2$ being an open set, consider $h(t,w)$, $(t,w)\in W$, that is holomorphic with respect to t (resp. continuous with repect to w). For all $\eta_1,\dots,\eta_n \in \mathbb{C}$ such that $(\eta_j,\eta_j)\in W$, $\forall\, j = 1,\dots,n$ and all $m_1,\dots,m_n \in \mathbb{N}$, we set

$$\begin{aligned}\mathcal{L}\left(\eta_1^{m_1},\dots,\eta_n^{m_n}; h(t,w)\right)(X) \quad &:= \\ := \quad \prod_{j=1}^{n}(X-\eta_j)^{m_j}\sum_{p=1}^{n}\frac{1}{(m_p-1)!}&\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p}\left(\frac{h(t,\eta_p)}{\prod_{j=1,j\neq p}^{n}(t-\eta_j)^{m_j}}\right).\end{aligned} \tag{1.5}$$

In particular, if $h(t)$ is holomorphic on $U\subset\mathbb{C}$ and $\eta_1,\dots,\eta_n\in U$, we set

$$\begin{aligned}\mathcal{L}\left(\eta_1^{m_1},\dots,\eta_n^{m_n}; h(t)\right)(X) \quad &:= \\ := \quad \prod_{j=1}^{n}(X-\eta_j)^{m_j}\sum_{p=1}^{n}\frac{1}{(m_p-1)!}&\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p}\left(\frac{h(t)}{\prod_{j=1,j\neq p}^{n}(t-\eta_j)^{m_j}}\right)\end{aligned} \tag{1.6}$$

(in the above sums only appear the terms with $m_p\geq 1$; if $m_p = 0$, $\forall\, p = 1,\dots,n$, we set by convention $\mathcal{L}(h) := 0$).

In addition, if we choose $\dfrac{h(t)}{X-t}$, then formula (1.6) gives the Lagrange interpolation polynomial of h on the $\{\eta_p\}_{1\leq p\leq n}$ with derivatives at order $\leq m_p-1$ (see section 3):

$$\begin{aligned}\mathcal{L}\left(\eta_1^{m_1},\dots,\eta_n^{m_n}\ ;\ \frac{h(t)}{X-t}\right)(X) \quad &= \\ = \quad \prod_{j=1}^{n}(X-\eta_j)^{m_j}\sum_{p=1}^{n}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p}&\left(\frac{h(t)}{(X-t)\prod_{j=1,j\neq p}^{n}(t-\eta_j)^{m_j}}\right) \\ = \quad \sum_{p=1}^{n}\ \prod_{j=1,j\neq p}^{n}(X-\eta_j)^{m_j}\ \sum_{s=0}^{m_p-1}(X-\eta_p)^s\,\frac{1}{s!}\frac{\partial^s}{\partial t^s}|_{t=\eta_p}&\left(\frac{h(t)}{\prod_{j=1,j\neq p}^{n}(t-\eta_j)^{m_j}}\right).\end{aligned}$$

In the particular case with $m_p = 1$, $\forall\, p = 1, \ldots, n$, we get the classical Lagrange interpolation polynomial

$$\mathcal{L}\left(\eta_1, \ldots, \eta_n \,;\, \frac{h(t)}{X-t}\right)(X) \quad = \quad \sum_{p=1}^{n} \prod_{j=1, j\neq p}^{n} \frac{X-\eta_j}{\eta_p-\eta_j}\, h(\eta_p)\,.$$

On the other hand, if $m_j = 0$, $\forall\, j \neq p$, we get the Taylor limited expansion on η_p at order $m_p - 1$:

$$\mathcal{L}\left(\eta_1^0, \ldots, \eta_p^{m_p}, \ldots, \eta_n^0 \,;\, \frac{h(t)}{X-t}\right)(X) \quad = \quad \sum_{s=0}^{m_p-1} (X-\eta_p)^s\, \frac{1}{s!}\, h^{(s)}(\eta_p)\,.$$

Next, for all $p = 1, \ldots, n$ and $u_p = 0, \ldots, m_p - 1$, we set

$$N_{u_p} \quad := \quad u_p + m_{p+1} + \cdots + m_n$$

and

$$N \quad := \quad m_1 + \cdots + m_n\,.$$

For $f \in \mathcal{O}(\mathbb{B}_2)$ and $f(z) = \sum_{k_1,k_2\geq 0} a_{k_1,k_2} z_1^{k_1} z_2^{k_2}$ its Taylor expansion, we set, for all $p = 1, \ldots, n-1$ and $u_p = 0, \ldots, m_p - 1$,

$$\begin{aligned}(1.7)\qquad R^0_{u_p}(f;z,t,w) \quad &:= \\ := \quad \frac{1+|w|^2\eta_p/t}{1+|w|^2} \sum_{k_1+k_2\geq N_{u_p}} a_{k_1,k_2}\, t^{k_1} &\left(\frac{z_2+|w|^2 z_1/t}{1+|w|^2}\right)^{k_1+k_2-N_{u_p}} z_2^{N_{u_p}}\end{aligned}$$

and

$$\begin{aligned}(1.8)\qquad R^0_N(f;z,t,w) \quad &:= \\ := \quad (z_1/z_2)^{m_1} \sum_{k_1+k_2\geq N} a_{k_1,k_2}\, t^{k_1-m_1} &\left(\frac{z_2+|w|^2 z_1/t}{1+|w|^2}\right)^{k_1+k_2-N+1} z_2^{N-1}\,.\end{aligned}$$

For all $u_1 = 0, \ldots, m_1 - 1$,

$$(1.9)\qquad R^1_{u_1}(f;z,t) \quad := \quad \sum_{k_1\leq u_1, k_2\geq N_{u_1}} a_{k_1,k_2}\, t^{k_1}\, z_2^{k_1+k_2}$$

and

$$(1.10)\qquad R^1_N(f;z,t) \quad := \quad (z_1/z_2)^{m_1} \sum_{k_1\leq m_1-1, k_2\geq N-k_1} a_{k_1,k_2}\, t^{k_1-m_1}\, z_2^{k_1+k_2}\,.$$

Lastly, for all $p = 2, \ldots, n-1$ and $u_p = 0, \ldots, m_p - 1$, we set

$$(1.11)\qquad R^2_{u_p}(f;z,t) \;:=\; \eta_p \sum_{k_2\leq m_n-1, k_1\geq N_{u_p}-k_2} a_{k_1,k_2}\, t^{N_{u_p}-1-k_2}\, z_1^{k_1+k_2-N_{u_p}}\, z_2^{N_{u_p}}$$

and

$$\begin{aligned}(1.12)\qquad R^2_N(f;z,t) \quad &:= \\ := \quad (z_1/z_2)^{m_1} &\sum_{k_2\leq m_n-1, k_1\geq N-k_2} a_{k_1,k_2}\, t^{N-m_1-1-k_2}\, z_1^{k_1+k_2-N+1}\, z_2^{N-1}\,.\end{aligned}$$

All these functions are well-defined since the above series are absolutely convergent for all $z \in \mathbb{B}_2$ and all (t,w) in a neighborhood of (η_p, η_p), $p = 1\ldots, n-1$ (see lemma 12).

We can finally set

$$\mathcal{G}\left(\eta_1^{m_1}, \ldots, \eta_n^{m_n}; f\right)(z) \quad := \tag{1.13}$$

$$\begin{aligned}
:= \quad & \sum_{u_1=0}^{m_1-1} (z_1/z_2)^{u_1} \mathcal{L}\left(\eta_2^{m_2}, \ldots, \eta_{n-1}^{m_{n-1}}; \frac{R^0_{u_1}(f;z,t,w) - R^1_{u_1}(f;z,t)}{t^{u_1+1}}\right)(z_1/z_2) \\
& + \sum_{p=2}^{n-1} \sum_{u_p=0}^{m_p-1} \mathcal{L}\left(\eta_p^{u_p+1}, \ldots, \eta_{n-1}^{m_{n-1}}; \frac{R^0_{u_p}(f;z,t,w) - R^2_{u_p}(f;z,t)}{z_1/z_2 - \eta_p}\right)(z_1/z_2) \\
& + \mathcal{L}\left(0^{m_n}; \frac{f(z_1,t)}{z_2 - t}\right)(z_2) \\
& - \mathcal{L}\left(\eta_2^{m_2}, \ldots, \eta_{n-1}^{m_{n-1}}; \frac{R^0_N(f;z,t,w) - R^1_N(f;z,t) - R^2_N(f;z,t)}{z_1/z_2 - t}\right)(z_1/z_2)\ .
\end{aligned}$$

Now we can give the principal result of this paper.

Theorem 1. *Let be $f \in \mathcal{O}(\mathbb{B}_2)$ and $f(z) = \sum_{k_1,k_2 \geq 0} a_{k_1,k_2}\, z_1^{k_1}\, z_2^{k_2}$ its Taylor expansion. Then for all $z \in \mathbb{B}_2$, we have*

$$f(z) \quad = \quad \mathcal{G}\left(\eta_1^{m_1}, \ldots, \eta_n^{m_n}; f\right)(z) \ + \sum_{k_1+k_2 \geq N,\, k_1 \geq m_1,\, k_2 \geq m_n} a_{k_1,k_2}\, z_1^{k_1}\, z_2^{k_2}\ .$$

The proof of this result will consist on the calculation of $Res(f, g_n)$ (section 5, proposition 4) and $PV(f, g_n)$ (section 4, proposition 3) from (1.1). As we will see, $Res(f, g_n)$ is a holomorphic function that is constructed from the data $f_{\{g_n(z)=0\}}$ and that interpolates f on $\{g_n(z) = 0\}$ (i.e. it coincides with f with derivatives on the lines $\{z_1 = \eta_p z_2\}$), while $PV(f, g_n)$ is the sum of a part that is constructed from $f_{\{g_n(z)=0\}}$ and of the above remainder that appears in the statement of the theorem.

It follows that any holomorphic function can be approximated on the unit ball by the explicit formula $\mathcal{G}\left(\eta_1^{m_1}, \ldots, \eta_n^{m_n}; f\right)$ that is constructed with the restrictions of f and its derivatives on the complex lines $\{z_1 - \eta_j z_2 = 0\}$, $j = 1, \ldots, n-1$ and $\{z_2 = 0\}$ (see [1], [11] for analogous results).

We see in addition that it is sufficient to assume that $f \in \mathcal{O}(\mathbb{B}_2)$. In particular, it is not necessary to assume that f is holomorphic on a neighborhood of the closed unit ball (condition that we need in order to construct interpolation functions with residual currents).

Although $\mathcal{G}\left(\eta_1^{m_1}, \ldots, \eta_n^{m_n}; f\right)$ is an explicit formula (see section 6, lemma 13), it is quite technical in the general case. Nevertheless, one can entirely specify it in

the special but not less natural case of single lines (i.e. $m_2 = \cdots = m_{n-1} = 1$ and $m_1 = m_n = 0$) where theorem 1 becomes:

$$f(z) \quad = \quad \mathcal{G}(\eta_2, \ldots, \eta_{n-1}; f)(z) + \sum_{k_1+k_2 \geq N} a_{k_1,k_2} z_1^{k_1} z_2^{k_2} \ ,$$

with

$$\mathcal{G}(\eta_2, \ldots, \eta_{n-1}; f)(z) \quad := \quad \mathcal{G}\left(\eta_1^0, \eta_2^1, \ldots, \eta_{n-1}^1, \eta_n^0; f\right)(z) \tag{1.14}$$

$$= \sum_{p=2}^{n-1} \prod_{j=p+1}^{n-1} (z_1 - \eta_j z_2) \ \times$$
$$\times \sum_{q=p}^{n-1} \frac{1+\eta_p \overline{\eta_q}}{1+|\eta_q|^2} \frac{1}{\prod_{j=p, j\neq q}^{n-1} (\eta_q - \eta_j)} \sum_{l \geq N-p+1} \left(\frac{z_2 + \overline{\eta_q} z_1}{1+|\eta_q|^2} \right)^{l-(N-p+1)} \frac{1}{l!} \frac{\partial^l}{\partial v^l}|_{v=0} [f(\eta_q v, v)]$$
$$- \sum_{p=2}^{n-1} \prod_{j=2, j\neq p}^{n-1} \frac{z_1 - \eta_j z_2}{\eta_p - \eta_j} \sum_{l \geq N} \left(\frac{z_2 + \overline{\eta_p} z_1}{1+|\eta_p|^2} \right)^{l-N+1} \frac{1}{l!} \frac{\partial^l}{\partial v^l}|_{v=0} [f(\eta_p v, v)] \ .$$

We finish with giving as a consequence of the theorem some information for the precision of this approximation (section 6, corollary 1). $\mathcal{F}$ being a compact subset of $\mathcal{O}(\mathbb{B}_2)$ (i.e. a subset of holomorphic functions that is uniformly bounded on any compact subset of the ball), one has for all compact subset $K \subset \mathbb{B}_2$

$$\sup_{f \in \mathcal{F}} \sup_{z \in K} \left| f(z) - \mathcal{G}\left(\eta_1^{m_1}, \ldots, \eta_n^{m_n}; f\right)(z) \right| \quad \leq \quad C(K, \mathcal{F})(1-\varepsilon_K)^N \ , \tag{1.15}$$

where $C(K, \mathcal{F})$ (resp. ε_K) depends on K and $\mathcal{F}$ (resp. K).

In conclusion some questions follow. Can the above precision be made better ? Or can it be made better with another choice of formula ? Since $\mathcal{G}(\eta_1^{m_1}, \ldots \eta_n^{m_n}; f)$ is linear with respect to the data $f_{\{g_n=0\}}$, theorem 1 can also be interpreted by the approximation of all 2-variable holomorphic function f by a fixed linear superposition of functions f_j of fewer variables (that depend on f). We know that one cannot get such an exact representation of all function f (see [13], [14]). Therefore we would like to get some lower bounds for the approximation of any compact subset of holomorphic functions by such a given family. We are also motivated in dealing with the approximation by a nonlinear family with respect to the data $f_{\{g_n=0\}}$ and getting lower bounds that only depend on the ε-entropy of $\mathcal{O}(\mathbb{B}_2)$ (see [10], [8], [9]).

Other questions follow: can we extend theorem 1 to the case of general analytic subvarieties (and not only for complex lines) ? Is the assertion still true with a general domain (and not only the ball) ? Finally, can we get analogous results in $\mathbb{C}^m$?

I would like to thank G. Henkin for the interesting discussions about this problem.

2. A preliminar formula in the general case

2.1. **General case.** We recall for $\zeta \in \mathbb{C}^m$

$$\begin{cases} \omega'(\zeta) = \sum_{k=1}^m (-1)^{k-1}\zeta_k \bigwedge_{j=1,j\neq k}^m d\zeta_k\,, \\ \omega(\zeta) \;=\; \bigwedge_{j=1}^m d\zeta_j \end{cases}$$

(as well as $d\omega'(\zeta) = m\omega(\zeta)$). The orientation on $\mathbb{C}^m$ is such that

$$\left(\frac{1}{i}\right)^{m^2} \int_{\zeta\in\mathbb{C}^m} \omega(\overline{\zeta}) \wedge \omega(\zeta) \quad > \quad 0\,.$$

Now we consider a bounded convex domain in $\mathbb{C}^m$

$$\Omega \quad = \quad \{z \in \mathbb{C}^m,\ \rho(z) < 0\}$$

with smooth boundary $\partial\Omega = \{z \in \mathbb{C}^m,\ \rho(z) = 0\}$ and whose orientation satisfies the Stokes formula on $\overline{\Omega}$ (this orientation is also chosen such that $\frac{1}{i^{m^2}}\int_{\zeta\in\partial\Omega} \omega'(\overline{\zeta}) \wedge \omega(\zeta) > 0$).

Let be $f,\, g \in \mathcal{O}(\overline{\Omega})$ (i.e. $\exists\, U \supset \overline{\Omega}$ open set such that $f,\, g \in \mathcal{O}(U)$) with $g \neq 0$ and $P = (P_1, \ldots, P_m) \in \mathcal{O}^m\left(\overline{\Omega}\times\overline{\Omega}\right)$ that satisfies: for all $(\zeta, z) \in \overline{\Omega}\times\overline{\Omega}$,

$$\begin{aligned} g(\zeta) - g(z) \quad &= \quad < P(\zeta, z), \zeta - z > \\ &= \quad \sum_{j=1}^m P_j(\zeta, z)\,(\zeta_j - z_j)\,. \end{aligned} \tag{2.1}$$

Lastly, we set

$$\varphi(\zeta, z) \quad = \quad < \frac{\partial\rho}{\partial z}(\zeta, z), \zeta - z > \ . \tag{2.2}$$

Now we can give the following preliminar result that gives the essential method to prove theorem 1.

Proposition 1. *For all $z \in \Omega$, we have*

$$\begin{aligned} (2.3)\ (-1)^{m(m-1)/2}\, f(z) \ &= \ \lim_{\varepsilon\to 0} g(z)\,\frac{(m-1)!}{(2\pi i)^m}\int_{\{\zeta\in\partial\Omega,\,|g(\zeta)|>\varepsilon\}} \frac{f(\zeta)\ \omega'\left(\frac{\partial\rho}{\partial z}\right)\wedge\omega(\zeta)}{g(\zeta)\ \varphi(\zeta, z)^m} \\ &+ \lim_{\varepsilon\to 0} \frac{(m-2)!}{(2\pi i)^m}\int_{\{\zeta\in\partial\Omega,|g(\zeta)|=\varepsilon\}} \frac{f(\zeta)\sum_{1\leq k<l\leq m}(-1)^{k+l-1}\left(\frac{\partial\rho}{\partial z_k}P_l - \frac{\partial\rho}{\partial z_l}P_k\right)\bigwedge_{j\neq k,l} d\left(\frac{\partial\rho}{\partial z_j}\right)\wedge\omega(\zeta)}{g(\zeta)\ \varphi(\zeta, z)^{m-1}}\,. \end{aligned}$$

Proof. For all $\varepsilon > 0$ and $z \in \Omega$, we consider the following differential form

$$\psi(\zeta, \lambda) \quad = \quad \frac{(m-1)!}{(2\pi i)^m}\ f(\zeta)\ \omega'\left(\frac{\lambda}{\varphi(\zeta, z)}\frac{\partial\rho}{\partial z}(\zeta) + (1-\lambda)\frac{P(\zeta, z)}{g(\zeta)}\right)\wedge\omega(\zeta)$$

that is defined on a neighborhood of

$$\Sigma_\varepsilon \quad = \quad \{\zeta \in \partial\Omega,\ |g(\zeta)| \geq \varepsilon\} \times [0, 1]\,,$$

with induced orientation from the one of $\Sigma_0 = \partial\Omega \times [0,1]$ that satisfies for all differential form $\chi_1(\zeta)$ and all function $\chi_2(\lambda)$

$$\int_{\{(\zeta,\lambda)\in\Sigma_0\}} \chi_1(\zeta)\wedge\chi_2(\lambda)d\lambda \;=\; \int_{\{\zeta\in\partial\Omega\}} \chi_1(\zeta) \times \int_0^1 \chi_2(\lambda)d\lambda\,.$$

The application of the Stokes formula gives

$$\int_{\partial\Sigma_\varepsilon} \psi(\zeta,\lambda) \;=\; \int_{\Sigma_\varepsilon} d\psi(\zeta,\lambda)\,, \tag{2.4}$$

with the associate orientation of $\partial\Sigma_\varepsilon$ that is specified in the following lemma.

Lemma 1. *We have*

$$\partial\Sigma_\varepsilon \;=\; -\{\zeta\in\partial\Omega,\,|g(\zeta)|=\varepsilon\}\times[0,1] - \{\zeta\in\partial\Omega,\,|g(\zeta)|\geq\varepsilon\}\times\{1\} + \{\zeta\in\partial\Omega,\,|g(\zeta)|\geq\varepsilon\}\times\{0\}\,.$$

It follows that

$$\int_{\partial\Sigma_\varepsilon}\psi(\zeta,\lambda) \;=\; -\int_{\{|g(\zeta)|=\varepsilon\}\times[0,1]}\psi(\zeta,\lambda) - \int_{\{|g(\zeta)|>\varepsilon\}}\psi(\zeta,1) + \int_{\{|g(\zeta)|>\varepsilon\}}\psi(\zeta,0)\,,$$

where the orientation of $\{\zeta\in\partial\Omega,\,|g(\zeta)|=\varepsilon\}\times[0,1]$ is defined, for all differential form $\chi_1(\zeta)$ and all function $\chi_2(\lambda)$, as

$$\int_{\{\zeta\in\partial\Omega,\,|g(\zeta)|=\varepsilon\}\times[0,1]}\chi_1(\zeta)\wedge\chi_2(\lambda)d\lambda \;=\; \int_{\{|g(\zeta)|=\varepsilon\}}\chi_1(\zeta)\times\int_0^1\chi_2(\lambda)d\lambda\,,$$

and the orientation of $\{\zeta\in\partial\Omega,\,|g(\zeta)|=\varepsilon\}$ is the one that satisfies the Stokes formula on $\{\zeta\in\partial\Omega,\,|g(\zeta)|\leq\varepsilon\}$.

Proof. First notice that, for all small enough $\varepsilon>0$, $\{\zeta\in\partial\Omega,\,|g(\zeta)|>\varepsilon\}$ (resp. $\{\zeta\in\partial\Omega,\,|g(\zeta)|=\varepsilon\}$) is a $(2m-1)$-dimensional (resp. $(2m-2)$-dimensional) submanifold.

Now consider $\chi(\zeta,\lambda)$ a differential form. It can be written as

$$\chi(\zeta,\lambda) \;=\; \chi_0(\zeta,\lambda)+\chi_1(\zeta,\lambda)\wedge d\lambda$$

where χ_0, χ_1 are differential forms of degree zero with respect to λ.

For $j=0,1$, one has

$$\int_{\{|g(\zeta)|>\varepsilon\}\times\{j\}}\chi(\zeta,\lambda) \;=\; \int_{\{|g(\zeta)|>\varepsilon\}}\chi_0(\zeta,j) = \int_{\{|g(\zeta)|>\varepsilon\}}\chi_{0,2m-1}(\zeta,j)\,, \tag{2.5}$$

where $\chi_{0,2m-1}(\zeta,j) := \sum_{k+l=2m-1}\sum_{|K|=k,|L|=l}\chi_{0,K,L}(\zeta,\lambda)d\zeta_K\wedge d\overline{\zeta}_L$ is the $(2m-1)$-homogeneous part of χ_0 with respect to ζ.

Similarly,

$$\begin{aligned}\int_{\{|g(\zeta)|=\varepsilon\}\times[0,1]}\chi(\zeta,\lambda) \;&=\; \int_{\{|g(\zeta)|=\varepsilon\}\times[0,1]}\chi_{1,2m-2}(\zeta,\lambda)\wedge d\lambda\\ &=\; \int_0^1 d\lambda\int_{\{|g(\zeta)|=\varepsilon\}}\chi_{1,2m-2}(\zeta,\lambda)\,.\end{aligned} \tag{2.6}$$

On the other hand,

$$
\begin{aligned}
\int_{\Sigma_\varepsilon} d\chi(\zeta,\lambda) &= \int_{\{|g(\zeta)|>\varepsilon\}\times[0,1]} d_\lambda(\chi_{0,2m-1}(\zeta,\lambda)) + d_\zeta(\chi_{1,2m-2}(\zeta,\lambda))\wedge d\lambda \\
&= \int_{\{|g(\zeta)|>\varepsilon\}\times[0,1]} \left((-1)^{2m-1}\frac{\partial\chi_{0,2m-1}}{\partial\lambda}(\zeta,\lambda) + d_\zeta(\chi_{1,2m-2}(\zeta,\lambda))\right)\wedge d\lambda \\
&= -\int_{\{|g(\zeta)|>\varepsilon\}}\int_0^1 d\lambda\,\frac{\partial\chi_{0,2m-1}}{\partial\lambda}(\zeta,\lambda) + \int_0^1 d\lambda\int_{\{|g(\zeta)|>\varepsilon\}} d_\zeta(\chi_{1,2m-2}(\zeta,\lambda)) \\
&= -\int_{\{|g(\zeta)|>\varepsilon\}} (\chi_{0,2m-1}(\zeta,1)-\chi_{0,2m-1}(\zeta,0)) - \int_0^1 d\lambda\int_{\{|g(\zeta)|=\varepsilon\}} \chi_{1,2m-2}(\zeta,\lambda)\,.
\end{aligned}
$$

The lemma follows by (2.4), (2.5) and (2.6).

$\surd$

Now we have $\psi(\zeta,0)=0$ since $P(\zeta,z)$ and $g(\zeta)$ are holomorphic.
Next, we claim that

$$
\psi(\zeta,1) = \frac{(m-1)!}{(2\pi i)^m}\, f(\zeta)\,\frac{\omega'\left(\frac{\partial\rho}{\partial z}(\zeta)\right)\wedge\omega(\zeta)}{\varphi(\zeta,z)^m}\,. \tag{2.7}
$$

Indeed, we have (since $d(\frac{1}{\varphi})\wedge d(\frac{1}{\varphi})=0$)

$$
\begin{aligned}
\omega'\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z}(\zeta)\right) &= \sum_{k=1}^m (-1)^{k-1}\frac{1}{\varphi}\frac{\partial\rho}{\partial z_k}\bigwedge_{j\neq k}\left(\frac{1}{\varphi}d\left(\frac{\partial\rho}{\partial z_j}\right)+\frac{\partial\rho}{\partial z_j}d\left(\frac{1}{\varphi}\right)\right) \\
&= \sum_{k=1}^m (-1)^{k-1}\frac{1}{\varphi}\frac{\partial\rho}{\partial z_k}\bigwedge_{j\neq k}\frac{1}{\varphi}d\left(\frac{\partial\rho}{\partial z_j}\right) \\
&\quad + \sum_{k=1}^m (-1)^{k-1}\frac{1}{\varphi}\frac{\partial\rho}{\partial z_k}\sum_{l\neq k}\bigwedge_{j<l,j\neq k}\frac{1}{\varphi}d\left(\frac{\partial\rho}{\partial z_j}\right)\wedge\frac{\partial\rho}{\partial z_l}d\left(\frac{1}{\varphi}\right)\wedge\bigwedge_{j>l,j\neq k}\frac{1}{\varphi}d\left(\frac{\partial\rho}{\partial z_j}\right)
\end{aligned}
$$

and the last sum is

$$
\begin{aligned}
&= \sum_{1\le l<k\le m}(-1)^{k-1}\frac{1}{\varphi}\frac{\partial\rho}{\partial z_k}(-1)^{l-1}\frac{\partial\rho}{\partial z_l}d\left(\frac{1}{\varphi}\right)\wedge\bigwedge_{j\neq k,l}\frac{1}{\varphi}d\left(\frac{\partial\rho}{\partial z_j}\right) \\
&\quad + \sum_{1\le k<l\le m}(-1)^{k-1}\frac{1}{\varphi}\frac{\partial\rho}{\partial z_k}(-1)^{l-2}\frac{\partial\rho}{\partial z_l}d\left(\frac{1}{\varphi}\right)\wedge\bigwedge_{j\neq k,l}\frac{1}{\varphi}d\left(\frac{\partial\rho}{\partial z_j}\right) \\
&= \sum_{1\le l<k\le m}(-1)^{k+l}\frac{1}{\varphi}\frac{\partial\rho}{\partial z_k}\frac{\partial\rho}{\partial z_l}d\left(\frac{1}{\varphi}\right)\wedge\bigwedge_{j\neq k,l}\frac{1}{\varphi}d\left(\frac{\partial\rho}{\partial z_j}\right) \\
&\quad - \sum_{1\le l<k\le m}(-1)^{k+l}\frac{1}{\varphi}\frac{\partial\rho}{\partial z_k}\frac{\partial\rho}{\partial z_l}d\left(\frac{1}{\varphi}\right)\wedge\bigwedge_{j\neq k,l}\frac{1}{\varphi}d\left(\frac{\partial\rho}{\partial z_j}\right) \\
&= 0\,,
\end{aligned}
$$

and this proves (2.7). It follows that

$$\int_{\{|g(\zeta)|>\varepsilon} \psi(\zeta,1) \quad = \quad \frac{(m-1)!}{(2\pi i)^m}\int_{\{|g(\zeta)|>\varepsilon} f(\zeta)\, \frac{\omega'\left(\frac{\partial\rho}{\partial z}(\zeta)\right)\wedge\omega(\zeta)}{\varphi(\zeta,z)^m}$$

then

$$\lim_{\varepsilon\to 0}\int_{\{|g(\zeta)|>\varepsilon\}} \psi(\zeta,1) \quad = \quad \frac{(m-1)!}{(2\pi i)^m}\int_{\zeta\in\partial\Omega} f(\zeta)\, \frac{\omega'\left(\frac{\partial\rho}{\partial z}(\zeta)\right)\wedge\omega(\zeta)}{\varphi(\zeta,z)^m} \tag{2.8}$$

$$= \quad (-1)^{m(m-1)/2}\, f(z)\,,$$

by the Cauchy-Fantappie formula (see [12]).

Now we want to specify $\psi(\zeta,\lambda)$ on $\{|g(\zeta)|=\varepsilon\}\times[0,1]$:

$$\omega'\left(\frac{P}{g}+\lambda\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z}-\frac{P}{g}\right)\right)\wedge\omega(\zeta) \quad =$$

$$\begin{aligned}
&= \sum_{k=1}^{m}(-1)^{k-1}\left(\frac{P_k}{g}+\lambda\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z_k}-\frac{P_k}{g}\right)\right)\bigwedge_{j\neq k}\left(\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z_j}-\frac{P_j}{g}\right)d\lambda+\lambda d\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z_j}\right)\right)\wedge\omega(\zeta)\\
&= \sum_{k<l}(-1)^{k-1}\left(\frac{P_k}{g}+\lambda\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z_k}-\frac{P_k}{g}\right)\right)(-1)^{2m-l}\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z_l}-\frac{P_l}{g}\right)\bigwedge_{j\neq k,l}\left(\lambda d\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z_j}\right)\right)\wedge\omega(\zeta)\wedge d\lambda\\
&+ \sum_{k>l}(-1)^{k-1}\left(\frac{P_k}{g}+\lambda\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z_k}-\frac{P_k}{g}\right)\right)(-1)^{2m-l-1}\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z_l}-\frac{P_l}{g}\right)\bigwedge_{j\neq k,l}\left(\lambda d\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z_j}\right)\right)\wedge\omega(\zeta)\wedge d\lambda\\
&= \sum_{k<l}(-1)^{k+l}\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z_k}\frac{P_l}{g}-\frac{1}{\varphi}\frac{\partial\rho}{\partial z_l}\frac{P_k}{g}\right)\bigwedge_{j\neq k,l}\left(\frac{1}{\varphi}d\left(\frac{\partial\rho}{\partial z_j}\right)+\frac{\partial\rho}{\partial z_j}d\left(\frac{1}{\varphi}\right)\right)\wedge\omega(\zeta)\wedge\lambda^{m-2}d\lambda\,.
\end{aligned}$$

Similarly, we see that, for all $1\le k<l\le m$,

$$\bigwedge_{j\neq k,l}\left(\frac{1}{\varphi}d\left(\frac{\partial\rho}{\partial z_j}\right)+\frac{\partial\rho}{\partial z_j}d\left(\frac{1}{\varphi}\right)\right) \quad =$$

$$\begin{aligned}
&= \frac{1}{\varphi^{m-2}}\bigwedge_{j\neq k,l} d\left(\frac{\partial\rho}{\partial z_j}\right)\\
&\quad + \frac{1}{\varphi^{m-3}}d\left(\frac{1}{\varphi}\right)\wedge\left[\sum_{u<k<l}(-1)^{u-1}+\sum_{k<u<l}(-1)^{u-2}+\sum_{k<l<u}(-1)^{u-3}\right]\frac{\partial\rho}{\partial z_u}\bigwedge_{j\neq k,l,u} d\left(\frac{\partial\rho}{\partial z_j}\right)\,,
\end{aligned}$$

and

$$\sum_{k<l}(-1)^{k+l}\left(\frac{\partial\rho}{\partial z_k}\frac{P_l}{g}-\frac{\partial\rho}{\partial z_l}\frac{P_k}{g}\right)\left[\sum_{u<k<l}(-1)^{u-1}+\sum_{k<u<l}(-1)^{u-2}+\sum_{k<l<u}(-1)^{u-3}\right]\frac{\partial\rho}{\partial z_u}\bigwedge_{j\neq k,l,u} d\left(\frac{\partial\rho}{\partial z_j}\right) =$$

$$
\begin{aligned}
&= \sum_{u<k<l} (-1)^{k+l+u} \bigwedge_{j\neq k,l,u} d\left(\frac{\partial\rho}{\partial z_j}\right) \times \\
&\quad \times \left[-\left(\frac{\partial\rho}{\partial z_k}\frac{P_l}{g} - \frac{\partial\rho}{\partial z_l}\frac{P_k}{g}\right)\frac{\partial\rho}{\partial z_u} + \left(\frac{\partial\rho}{\partial z_u}\frac{P_l}{g} - \frac{\partial\rho}{\partial z_l}\frac{P_u}{g}\right)\frac{\partial\rho}{\partial z_k} - \left(\frac{\partial\rho}{\partial z_u}\frac{P_k}{g} - \frac{\partial\rho}{\partial z_k}\frac{P_u}{g}\right)\frac{\partial\rho}{\partial z_l}\right] \\
&= 0\,.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\omega'\left(\frac{P}{g} + \lambda\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z} - \frac{P}{g}\right)\right) \wedge \omega(\zeta) \quad &= \\
&= \frac{1}{\varphi^{m-1}} \sum_{k<l} (-1)^{k+l} \left(\frac{\partial\rho}{\partial z_k}\frac{P_l}{g} - \frac{\partial\rho}{\partial z_l}\frac{P_k}{g}\right) \bigwedge_{j\neq k,l} d\left(\frac{\partial\rho}{\partial z_j}\right) \wedge \omega(\zeta) \wedge \lambda^{m-2} d\lambda
\end{aligned}
$$

then

$$
\begin{aligned}
(2.9)\quad \int_{\{|g(\zeta)|=\varepsilon\}\times[0,1]} \psi(\zeta,\lambda) \quad &= \\
&= \frac{(m-1)!}{(2\pi i)^m} \int_{\{|g(\zeta)|=\varepsilon\}} \frac{f(\zeta)}{\varphi^{m-1}} \sum_{k<l} (-1)^{k+l} \left(\frac{\partial\rho}{\partial z_k}\frac{P_l}{g} - \frac{\partial\rho}{\partial z_l}\frac{P_k}{g}\right) \bigwedge_{j\neq k,l} d\left(\frac{\partial\rho}{\partial z_j}\right) \wedge \omega(\zeta) \times \int_0^1 \lambda^{m-2} d\lambda \\
&= \frac{(m-2)!}{(2\pi i)^m} \int_{\{|g(\zeta)|=\varepsilon\}} \frac{f(\zeta)\sum_{k<l}(-1)^{k+l}\left(\frac{\partial\rho}{\partial z_k}(\zeta)P_l(\zeta,z) - \frac{\partial\rho}{\partial z_l}(\zeta)P_k(\zeta,z)\right)}{g(\zeta)\ \varphi(\zeta,z)^{m-1}} \bigwedge_{j\neq k,l} d\left(\frac{\partial\rho}{\partial z_j}(\zeta)\right) \wedge \omega(\zeta)\,.
\end{aligned}
$$

Finally, we want to specify $d\psi(\zeta,\lambda)$ on $\{|g(\zeta)| > \varepsilon\} \times [0,1]$. Since f, g and P are holomorphic, one has

$$
\begin{aligned}
d\psi(\zeta,\lambda) &= \frac{m!}{(2\pi i)^m}\, f(\zeta)\, \omega\left(\frac{\lambda}{\varphi(\zeta,z)}\frac{\partial\rho}{\partial z}(\zeta) + (1-\lambda)\frac{P(\zeta,z)}{g(\zeta)}\right) \wedge \omega(\zeta) \\
&= \frac{m!}{(2\pi i)^m}\, f(\zeta)\, \omega\left(\lambda\left(\frac{1}{\varphi(\zeta,z)}\frac{\partial\rho}{\partial z}(\zeta) - \frac{P(\zeta,z)}{g(\zeta)}\right)\right) \wedge \omega(\zeta)
\end{aligned}
$$

and

$$
\begin{aligned}
\omega\left(\lambda\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z}-\frac{P}{g}\right)\right)\wedge\omega(\zeta) &= \bigwedge_{k=1}^{m}\left(\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z_k}-\frac{P_k}{g}\right)d\lambda+\lambda d\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z_k}\right)\right)\wedge\omega(\zeta)\\
&= \sum_{k=1}^{m}(-1)^{2m-k}\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z_k}-\frac{P_k}{g}\right)\bigwedge_{l\neq k}d\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z_l}\right)\wedge\omega(\zeta)\wedge\lambda^{m-1}d\lambda\\
&= -\omega'\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z}\right)\wedge\omega(\zeta)\wedge\lambda^{m-1}d\lambda\\
&\quad+\sum_{k=1}^{m}(-1)^{k-1}\frac{P_k}{g}\bigwedge_{l\neq k}d\left(\frac{1}{\varphi}\frac{\partial\rho}{\partial z_l}\right)\wedge\omega(\zeta)\wedge\lambda^{m-1}d\lambda\,.\\
&= -\frac{1}{\varphi^m}\,\omega'\left(\frac{\partial\rho}{\partial z}\right)\wedge\omega(\zeta)\wedge\lambda^{m-1}d\lambda\\
&\quad+\frac{1}{\varphi^{m-1}}\sum_{k=1}^{m}(-1)^{k-1}\frac{P_k}{g}\bigwedge_{l\neq k}d\left(\frac{\partial\rho}{\partial z_l}\right)\wedge\omega(\zeta)\wedge\lambda^{m-1}d\lambda\\
&\quad+\frac{1}{\varphi^{m-2}}\,d\left(\frac{1}{\varphi}\right)\wedge\sum_{k=1}^{m}(-1)^{k-1}\frac{P_k}{g}\ \times\\
&\quad\times\ \left[\sum_{l<k}(-1)^{l-1}+\sum_{l>k}(-1)^{l-2}\right]\frac{\partial\rho}{\partial z_l}\bigwedge_{j\neq k,l}d\left(\frac{\partial\rho}{\partial z_j}\right)\wedge\omega(\zeta)\wedge\lambda^{m-1}d\lambda\,,
\end{aligned}
\tag{2.10}
$$

the last equality coming from (2.7). On the other hand, since $\varphi(\zeta,z)=\sum_{q=1}^{m}(\zeta_q-z_q)\frac{\partial\rho}{\partial z_q}$, one has

$$
\frac{1}{\varphi^{m-2}}d\left(\frac{1}{\varphi}\right)\wedge\sum_{k=1}^{m}(-1)^{k-1}\frac{P_k}{g}\left[\sum_{l<k}(-1)^{l-1}+\sum_{l>k}(-1)^{l}\right]\frac{\partial\rho}{\partial z_l}\bigwedge_{j\neq k,l}d\left(\frac{\partial\rho}{\partial z_j}\right)\wedge\omega(\zeta) =
$$

$$
\begin{aligned}
&= \frac{1}{\varphi^m}\sum_{q=1}^{m}(\zeta_q-z_q)d\left(\frac{\partial\rho}{\partial z_q}\right)\wedge\sum_{k<l}(-1)^{k+l}\left(\frac{P_k}{g}\frac{\partial\rho}{\partial z_l}-\frac{P_l}{g}\frac{\partial\rho}{\partial z_k}\right)\bigwedge_{j\neq k,l}d\left(\frac{\partial\rho}{\partial z_j}\right)\wedge\omega(\zeta)\\
&= \frac{1}{\varphi^m}\sum_{k<l}(-1)^{k+l}(\zeta_k-z_k)\left(\frac{P_k}{g}\frac{\partial\rho}{\partial z_l}-\frac{P_l}{g}\frac{\partial\rho}{\partial z_k}\right)(-1)^{k-1}\bigwedge_{j\neq l}d\left(\frac{\partial\rho}{\partial z_j}\right)\wedge\omega(\zeta)\\
&\quad+\frac{1}{\varphi^m}\sum_{k<l}(-1)^{k+l}(\zeta_l-z_l)\left(\frac{P_k}{g}\frac{\partial\rho}{\partial z_l}-\frac{P_l}{g}\frac{\partial\rho}{\partial z_k}\right)(-1)^{l-2}\bigwedge_{j\neq k}d\left(\frac{\partial\rho}{\partial z_j}\right)\wedge\omega(\zeta)\\
&= \frac{1}{\varphi^m}\sum_{k>l}(-1)^{k-1}(\zeta_l-z_l)\left(\frac{P_l}{g}\frac{\partial\rho}{\partial z_k}-\frac{P_k}{g}\frac{\partial\rho}{\partial z_l}\right)\bigwedge_{j\neq k}d\left(\frac{\partial\rho}{\partial z_j}\right)\wedge\omega(\zeta)\\
&\quad+\frac{1}{\varphi^m}\sum_{k<l}(-1)^{k}(\zeta_l-z_l)\left(\frac{P_k}{g}\frac{\partial\rho}{\partial z_l}-\frac{P_l}{g}\frac{\partial\rho}{\partial z_k}\right)\bigwedge_{j\neq k}d\left(\frac{\partial\rho}{\partial z_j}\right)\wedge\omega(\zeta)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\varphi^m}\sum_{k=1}^m(-1)^k\sum_{l=1}^m(\zeta_l-z_l)\left(\frac{P_k}{g}\frac{\partial\rho}{\partial z_l}-\frac{P_l}{g}\frac{\partial\rho}{\partial z_k}\right)\bigwedge_{j\neq k} d\left(\frac{\partial\rho}{\partial z_j}\right)\wedge\omega(\zeta)\\
&= \frac{1}{\varphi^m}\sum_{k=1}^m(-1)^k\left(\frac{P_k}{g}\,\varphi-\frac{g(\zeta)-g(z)}{g(\zeta)}\frac{\partial\rho}{\partial z_k}\right)\bigwedge_{j\neq k} d\left(\frac{\partial\rho}{\partial z_j}\right)\wedge\omega(\zeta)\\
&= \frac{1}{\varphi^{m-1}}\sum_{k=1}^m(-1)^k\frac{P_k}{g}\bigwedge_{j\neq k} d\left(\frac{\partial\rho}{\partial z_j}\right)\wedge\omega(\zeta)\ +\ \frac{1}{\varphi^m}\left(1-\frac{g(z)}{g(\zeta)}\right)\omega'\left(\frac{\partial\rho}{\partial z}\right)\wedge\omega(\zeta)\,.
\end{aligned}
$$

It follows from (2.10) that

$$
\begin{aligned}
\int_{\{|g(\zeta)|>\varepsilon\}\times[0,1]} d\psi(\zeta,\lambda) &= -\frac{m!}{(2\pi i)^m}\int_{\{|g(\zeta)|>\varepsilon\}}\frac{f(\zeta)\,g(z)}{g(\zeta)\,\varphi(\zeta,z)}\,\omega'\left(\frac{\partial\rho}{\partial z}\right)\wedge\omega(\zeta)\ \times\ \int_0^1\lambda^{m-1}d\lambda\\
&= -\,g(z)\,\frac{(m-1)!}{(2\pi i)^m}\int_{\{|g(\zeta)|>\varepsilon\}}\frac{f(\zeta)}{g(\zeta)\,\varphi(\zeta,z)}\,\omega'\left(\frac{\partial\rho}{\partial z}\right)\wedge\omega(\zeta)\,. \qquad (2.11)
\end{aligned}
$$

Lastly, we deduce by lemma 1, (2.4), (2.8), (2.9) and (2.11) that, for all $z\in\mathbb{B}_2$,

$$
\begin{aligned}
-\lim_{\varepsilon\to 0} g(z)\,\frac{(m-1)!}{(2\pi i)^m}\int_{\{|g(\zeta)|>\varepsilon\}}\frac{f(\zeta)}{g(\zeta)\,\varphi(\zeta,z)}\,\omega'\left(\frac{\partial\rho}{\partial z}\right)\wedge\omega(\zeta) \quad &= \\
= \ -\lim_{\varepsilon\to 0}\frac{(m-2)!}{(2\pi i)^m}\int_{\{|g(\zeta)|=\varepsilon\}}\frac{f(\zeta)\sum_{k<l}(-1)^{k+l}\left(\frac{\partial\rho}{\partial z_k}P_l-\frac{\partial\rho}{\partial z_l}P_k\right)}{g(\zeta)\ \varphi(\zeta,z)^{m-1}}&\bigwedge_{j\neq k,l} d\left(\frac{\partial\rho}{\partial z_j}\right)\wedge\omega(\zeta)\\
-\,(-1)^{m(m-1)/2}\,f(z)&
\end{aligned}
$$

and the proof of the proposition is achieved.

$\surd$

2.2. **Case of theorem 1.** Now consider $\mathbb{C}^2$ with $\Omega=\mathbb{B}_2$ and $\partial\Omega=\mathbb{S}_2=\{|z_1|^2+|z_2|^2=1\}$ where $\rho(\zeta)=\|\zeta\|^2-1=<\overline{\zeta},\zeta>-1$. Moreover,

$$
\begin{cases}
\frac{\partial\rho}{\partial z}(\zeta)=\overline{\zeta}\,,\\
\varphi(\zeta,z)=<\overline{\zeta},\zeta-z>=1-<\overline{\zeta},z>\ .
\end{cases}
$$

We also choose (see (1.2) in introduction)

$$
g_n(z) \quad = \quad z_1^{m_1}\prod_{j=2}^{n-1}(z_1-\eta_j z_2)^{m_j}\ z_2^{m_n}
$$

with $m_j\geq 0,\ j=1,\dots,n$,

$$
0=|\eta_1|<|\eta_2|\leq\cdots\leq|\eta_{n-1}|<|\eta_n|=+\infty
$$

and associate $P_n(\zeta,z)=(P_n^1(\zeta,z),P_n^2(\zeta,z))$ (we will specify $P_n(\zeta,z)$ in section 5). One could use proposition 1 in the following. Nevertheless, in order to prove theorem 1, we need another set than $\{|g_n(\zeta)|>\varepsilon\}$. For all $p=1,\dots,n$ we set

$$\alpha_p \quad := \quad \frac{|\eta_p|}{\sqrt{1+|\eta_p|^2}} \tag{2.12}$$

(with $\alpha_n:=1$). Then $0=\alpha_1<\alpha_2\leq\cdots\leq\alpha_{n-1}<\alpha_n$.

Now there are $\widetilde{n}\leq n$ and $1=q_1<\cdots<q_{\widetilde{n}}=n$ such that

$$\begin{aligned} 0=\alpha_1=\alpha_{q_1}<\alpha_{q_1+1} &= \cdots = \alpha_{q_2}<\cdots \\ <\alpha_{q_l+1} &= \cdots = \alpha_{q_{l+1}}<\cdots \\ <\alpha_{q_{\widetilde{n}-2}+1} &= \cdots = \alpha_{q_{\widetilde{n}-1}}<\alpha_{q_{\widetilde{n}}}=\alpha_n=1\,. \end{aligned} \tag{2.13}$$

Then for all small enough $\varepsilon>0$ we can set

$$\widetilde{\Sigma}_\varepsilon \quad := \quad \bigcup_{l=1}^{\widetilde{n}-1}\left\{\zeta\in\mathbb{S}_2,\ \alpha_{q_l}+\varepsilon<|\zeta_1|<\alpha_{q_{l+1}}-\varepsilon\right\} \tag{2.14}$$

with boundary $\partial\widetilde{\Sigma}_\varepsilon$ and orientation that satisfies Stokes formula. For the proof of theorem 1 we will use the following result that is similar to proposition 1 with $\widetilde{\Sigma}_\varepsilon$:

Proposition 2. *For all $f\in\mathcal{O}\left(\overline{\mathbb{B}_2}\right)$ and all $z\in\mathbb{B}_2$,*

$$\begin{aligned} f(z) \quad &= \\ = \quad &\lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2}\left[\int_{|\zeta_1|=1-\varepsilon}-\sum_{l=2}^{\widetilde{n}-1}\left(\int_{|\zeta_1|=\alpha_{q_l}+\varepsilon}-\int_{|\zeta_1|=\alpha_{q_l}-\varepsilon}\right)-\int_{|\zeta_1|=\varepsilon}\right]\frac{f(\zeta)\det\left(\overline{\zeta},P_n(\zeta,z)\right)}{g_n(\zeta)\left(1-<\overline{\zeta},z>\right)}\,\omega(\zeta) \\ &-\lim_{\varepsilon\to 0}\frac{g_n(z)}{(2\pi i)^2}\sum_{l=1}^{\widetilde{n}-1}\int_{\alpha_{q_l}+\varepsilon<|\zeta_1|<\alpha_{q_{l+1}}-\varepsilon}\frac{f(\zeta)\,\omega'\left(\overline{\zeta}\right)\wedge\omega(\zeta)}{g_n(\zeta)\ \left(1-<\overline{\zeta},z>\right)^2}\,. \end{aligned} \tag{2.15}$$

Proof. The proof is similar to the one of proposition 1 with

$$\widetilde{\psi}(\zeta,\lambda) \quad = \quad \frac{1}{(2\pi i)^2}\,f(\zeta)\,\omega'\left(\lambda\frac{\overline{\zeta}}{1-<\overline{\zeta},z>}+(1-\lambda)\frac{P_n(\zeta,z)}{g_n(\zeta)}\right)\wedge\omega(\zeta)\,.$$

In particular, we see that, for all small enough $\varepsilon>0$, $\widetilde{\psi}$ is well-defined on a neighborhood of $\overline{\widetilde{\Sigma}_\varepsilon}$. Indeed, if for $j=1,\dots,n-1$, $\zeta_1-\eta_j\zeta_2$ vanishes on $\mathbb{S}_2$, then

$$\begin{cases}|\zeta_1|=|\eta_j|\,|\zeta_2| \\ |\zeta_1|^2+|\zeta_2|^2=1\end{cases} \quad\Rightarrow\quad |\zeta_1|^2=|\eta_j|^2(1-|\zeta_1|^2)$$

thus $|\zeta_1|=\alpha_j$.

The only difference is to specify the orientation on $\widetilde{\Sigma}_\varepsilon$ and $\partial\widetilde{\Sigma}_\varepsilon$. First, the orientation on $\widetilde{\Sigma}_\varepsilon$ is induced by the one on $\mathbb{S}_2$. Next, for all $0<\beta_1<\beta_2<1$, we have as in lemma 1

$$\partial\left(\{\beta_1<|\zeta_1|<\beta_2\}\times[0,1]\right) \quad = \quad \left(\partial\{\beta_1<|\zeta_1|<\beta_2\}\right)\times[0,1]\,-\,\{\beta_1<|\zeta_1|<\beta_2\}\times(\partial[0,1])$$

where the orientation of $\partial\{\beta_1 < |\zeta_1| < \beta_2\}$ is fixed by the Stokes formula on $\{\beta_1 < |\zeta_1| < \beta_2\}$, i.e. for all $(2n-2)$-form χ,

$$\int_{\{\beta_1<|\zeta_1|<\beta_2\}} d\chi(\zeta) \quad = \quad \int_{\{|\zeta_1|=\beta_2\}} \chi(\zeta) - \int_{\{|\zeta_1|=\beta_1\}} \chi(\zeta)\,.$$

It follows that

$$\partial\widetilde{\Sigma}_\varepsilon \quad = \quad \sum_{l=1}^{\widetilde{n}-1} \big(\{|\zeta_1| = \alpha_{q_{l+1}} - \varepsilon\} \,-\, \{|\zeta_1| = \alpha_{q_l} + \varepsilon\}\big)\ .$$

Since

$$\lim_{\varepsilon\to 0} \int_{\widetilde{\Sigma}_\varepsilon} \psi(\zeta,1) \quad = \quad \frac{1}{(2\pi i)^2} \int_{\mathbb{S}_2} f(\zeta)\, \frac{\omega'(\overline{\zeta}) \wedge \omega(\zeta)}{(1- <\overline{\zeta}, z>^2} \;=\; -\,f(z)\,,$$

we get

$$-\lim_{\varepsilon\to 0} \frac{g_n(z)}{(2\pi i)^2} \sum_{l=1}^{\widetilde{n}-1} \int_{\alpha_{q_l}+\varepsilon<|\zeta_1|<\alpha_{q_{l+1}}-\varepsilon} \frac{f(\zeta)\,\omega'\left(\overline{\zeta}\right) \wedge \omega(\zeta)}{g_n(\zeta)\,\left(1- <\overline{\zeta}, z>\right)^2} \quad =$$

$$= \quad \lim_{\varepsilon\to 0} \sum_{l=1}^{\widetilde{n}-1} \frac{1}{(2\pi i)^2} \left[\int_{|\zeta_1|=\alpha_{q_{l+1}}-\varepsilon} - \int_{|\zeta_1|=\alpha_{q_l}+\varepsilon}\right] \frac{-f(\zeta)\,(\overline{\zeta}_1 P_n^2 - \overline{\zeta}_2 P_n^1)}{g_n(\zeta)\,(1- <\overline{\zeta}, z>)}\,\omega(\zeta) + f(z)$$

and the proposition follows.

$\surd$

3. Some preliminar results on the Lagrange interpolation formula

Let $W \subset \mathbb{C}$ be an open set, $f \in \mathcal{O}(W)$ and $\eta_1, \ldots, \eta_n \in W$ be different complex numbers (with associate multiplicities $m_1, \ldots, m_n \in \mathbb{N}$). Consider the following Lagrange interpolation polynomial for f on the η_j:

$$\text{(3.1)} \quad L_{f,\eta^m}(X) \quad := \quad \mathcal{L}\left(\eta_1^{m_1}, \ldots, \eta_n^{m_n}; \frac{f(t)}{X-t}\right)$$

$$= \quad \prod_{j=1}^{n} (X-\eta_j)^{m_j} \sum_{p=1}^{n} \frac{1}{(m_p-1)!} \frac{\partial^{m_p-1}}{\partial t^{m_p-1}}\Big|_{t=\eta_p} \left(\frac{f(t)}{(X-t)\prod_{j=1, j\neq p}^{n} (t-\eta_j)^{m_j}}\right)$$

$$= \quad \sum_{p=1}^{n} \prod_{j=1, j\neq p}^{n} (X-\eta_j)^{m_j} \sum_{s=0}^{m_p-1} (X-\eta_p)^s\, \frac{1}{s!} \frac{\partial^s}{\partial t^s}\Big|_{t=\eta_p} \left(\frac{f(t)}{\prod_{j=1, j\neq p}^{n} (t-\eta_j)^{m_j}}\right)\,.$$

One has the following preliminar result:

Lemma 2. *L_{f,η^m} is the unique polynomial $P \in \mathbb{C}[X]$ with degree at most $N-1$ (where $N = m_1 + \cdots + m_n$) that satisfies, for all $p = 1, \ldots, n$ and all $s = 1, \ldots, m_p - 1$,*

$$P^{(s)}(\eta_p) \quad = \quad f^{(s)}(\eta_p)\,.$$

Proof. First, we have for all $p = 1, \ldots, n$ and all $l = 1, \ldots, m_p - 1$,

$$\begin{aligned}
L^{(l)}_{f,\eta^m}(\eta_p) \quad &= \\
&= \frac{\partial^l}{\partial X^l}|_{X=\eta_p} \prod_{j=1, j\neq p}^{n} (X-\eta_j)^{m_j} \sum_{s=0}^{m_p-1} (X-\eta_p)^s \frac{1}{s!} \frac{\partial^s}{\partial t^s}|_{t=\eta_p} \left(\frac{f(t)}{\prod_{j=1,j\neq p}^{n}(t-\eta_j)^{m_j}} \right) \\
&= \sum_{u=0}^{l} \frac{l!}{u!\,(l-u)!} \frac{\partial^{l-u}}{\partial X^{l-u}}|_{X=\eta_p} \left(\prod_{j=1,j\neq p}^{n} (X-\eta_j)^{m_j} \right) \times \\
&\qquad \times u!\, \frac{1}{u!} \frac{\partial^u}{\partial t^u}|_{t=\eta_p} \left(\frac{f(t)}{\prod_{j=1,j\neq p}^{n}(t-\eta_j)^{m_j}} \right) \\
&= \frac{\partial^u}{\partial t^u}|_{t=\eta_p} \left[\prod_{j=1,j\neq p}^{n} (t-\eta_j)^{m_j} \times \frac{f(t)}{\prod_{j=1,j\neq p}^{n}(t-\eta_j)^{m_j}} \right] \\
&= f^{(l)}(\eta_p)\,.
\end{aligned}$$

Finally, let be $P \in \mathbb{C}[X]$ another polynomial of degree at most $N-1$ that satisfies $L^{(s)}_{f,\eta^m}(\eta_p) = f^{(s)}(\eta_p)$, for all $p = 1, \ldots, n$ and all $s = 0, \ldots, m_p - 1$. It follows that $P - L_{f,\eta^m}$ is divisible by $\prod_{j=1}^{n}(X-\eta_j)$ and of degree at most $N-1$ then $P - L_{f,\eta^m} = 0$ and this proves the lemma.

$\surd$

We have an additional result when f is a polynomial function.

Lemma 3. *Consider the Euclidean division of $P \in \mathbb{C}[X]$ with degree k by $G(X) := \prod_{j=1}^{n}(X-\eta_j)^{m_j}$*

$$P \quad = \quad G \cdot Q(P,G) \,+\, R(P,G)\,,$$

where $Q(P,G)$ (resp. $R(P,G)$) is the quotient (resp. remainder). Then

$$R(P,G) \quad = \quad L_{P,\eta^m}\,.$$

In particular,

$$\frac{P(X)}{G(X)} - \sum_{p=1}^{n} \frac{1}{(m_p-1)!} \frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p} \left[\frac{P(t)}{(X-t)\prod_{j=1,j\neq p}^{n}(t-\eta_j)^{m_j}} \right] = \tag{3.2}$$

$$= Q(P,G)\,.$$

If $P(X) = X^k$ we have in addition

$$Q(X^k, G) \quad = \quad \sum_{u=0}^{k-N} X^{k-N-u} \sum_{v_1+\cdots+v_n=u} \prod_{j=1}^{n} \frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!} \eta_j^{v_j}\,. \tag{3.3}$$

Proof. First, we have $G^{(s)}(\eta_p) = 0$ for all $p = 1, \dots, n$ and all $s = 0, \dots, m_p - 1$ then

$$R(P,G)^{(s)}(\eta_p) = P^{(s)}(\eta_p) = L^{(s)}_{P,\eta^m}(\eta_p)\,.$$

The first assertion follows since L_{P,η^m} and $R(P,G)$ have degree at most $N-1$.

Next, (3.2) follows from the Euclidean division of P by G.

Now consider $P(X) = X^k$. If $k < N$ then $Q(X^k, G) = 0$ and the second assertion is obvious. If $k \geq N$ we can write $G(X) = \prod_{j=1}^N (X - \eta'_j)$ (where the η'_j are not necessarily different) and we prove the assertion by induction on N.

If $N = 1$ we have

$$\begin{aligned} X^k &= X^k - {\eta'_1}^k + {\eta'_1}^k \\ &= (X - \eta'_1) \sum_{u=0}^{k-1} X^{k-1-u} {\eta'_1}^u + {\eta'_1}^k\,. \end{aligned}$$

Now assume that it is true for $N-1$ and let be $\eta'_N \in \mathbb{C}$. We have similarly

$$X^k = (X - \eta'_N) \sum_{v_N=0}^{k-1} {\eta'_N}^{v_N} X^{k-1-v_N} + {\eta'_N}^k$$

Since

$$\begin{aligned} Q\left(X^{k-1-v_N}, \prod_{j=1}^{N-1}(X - \eta'_j)\right) &= \sum_{w=0}^{k-N-v_N} X^{k-N-v_N-w} \sum_{v_1+\cdots+v_{N-1}=w} \prod_{j=1}^{N-1} {\eta'_j}^{v_j} \\ &= \sum_{u=v_N}^{k-N} X^{k-N-u} \sum_{v_1+\cdots+v_{N-1}=u-v_N} \prod_{j=1}^{N-1} {\eta'_j}^{v_j}\,, \end{aligned}$$

it follows that

$$\begin{aligned} X^k &= (X - \eta'_N) \prod_{j=1}^{N-1}(X - \eta'_j) \sum_{u=0}^{k-N} X^{k-N-u} \sum_{v_N=0}^{\min(k-1,u)} {\eta'_N}^{v_N} \sum_{v_1+\cdots+v_{N-1}=u-v_N} \prod_{j=1}^{N-1} {\eta'_j}^{v_j} \\ &\quad + (X - \eta'_N)R + {\eta'_N}^k \\ &= G(X) \sum_{u=0}^{k-N} X^{k-N-u} \sum_{v_N=0}^{u} \sum_{v_1+\cdots+v_{N-1}+v_N=u} {\eta'_N}^{v_N} \prod_{j=1}^{N-1} {\eta'_j}^{v_j} \;+\; (X - \eta'_N)R + {\eta'_N}^k\,, \end{aligned}$$

with $\deg R \leq N-2$ and this proves the assertion.

To finish the proof, we notice that for all $u = 0, \dots, k-N$

$$\begin{aligned}
\sum_{v_1+\cdots+v_N=u} \prod_{j=1}^{N} {\eta'_j}^{v_j} &= \sum_{v_{1,1}+\cdots+v_{n,m_n}=u} \prod_{l=1}^{n} \eta_l^{v_{l,1}+\cdots+v_{l,m_l}} \\
&= \sum_{w_1+\cdots+w_n=u} \prod_{l=1}^{n} \left(\eta_l^{w_l}\, card\{v_{l,1}+\cdots+v_{l,m_l}=w_l\}\right) \\
&= \sum_{w_1+\cdots+w_n=u} \prod_{l=1}^{n} \frac{(w_l+m_l-1)!}{w_l!\,(m_l-1)!}\,\eta_l^{w_l}\,,
\end{aligned}$$

the last equality coming from the following lemma.

$\surd$

Lemma 4. *For all $m \geq 1$ and $q \geq 0$,*

$$card\{(v_1,\dots,v_m)\in\mathbb{N}^m,\ v_1+\cdots+v_m=q\} \quad = \quad \frac{(q+m-1)!}{q!\,(m-1)!}\,.$$

Proof. Consider the following formal series

$$\sum_{v_1,\dots,v_m\geq 0} X_1^{v_1}\cdots X_m^{v_m}$$

The coefficient of order q after evaluation $X_1 = \cdots = X_m = X$ is exactly $card\{v_1+\cdots+v_m=q\}$. On the other hand, we have

$$\prod_{j=1}^{m}\left(\sum_{v_j\geq 0} X_j^{v_j}\right) = \prod_{j=1}^{m}\frac{1}{1-X_j}$$

that gives after evaluation

$$\begin{aligned}
\frac{1}{(1-X)^m} &= \frac{1}{(m-1)!}\frac{d^{m-1}}{dX^{m-1}}\left(\frac{1}{1-X}\right) \\
&= \frac{1}{(m-1)!}\sum_{k\geq m-1} k(k-1)\cdots(k-m+2)X^{k-m+1},
\end{aligned}$$

whose coefficient of order q is $\dfrac{(q+m-1)\cdots(q+1)}{(m-1)!}$.

$\surd$

4. Calculation of the remainder

We set

$$(4.1)\qquad U_\eta \;=\; \{z\in\mathbb{C}^2,\ z_1\neq 0,\ z_2\neq 0 \text{ and } \forall\, p=2,\dots,n-1,\ z_1-\eta_p z_2\neq 0\}\,.$$

We also remind $g_n(z) = z_1^{m_1}\prod_{j=2}^{n-1}(z_1-\eta_j z_2)^{m_j} z_2^{m_n},\ m_1,\dots,m_n\in\mathbb{N}$. So we can give the following result that we will prove in this section.

Proposition 3. *For all $z \in U_\eta$, we have*

$$(4.2)\quad -\lim_{\varepsilon\to 0}\frac{g_n(z)}{(2\pi i)^2}\int_{\Sigma_\varepsilon}\frac{\zeta_1^{k_1}\zeta_2^{k_2}\ \omega'\left(\overline{\zeta}\right)\wedge\omega(\zeta)}{g_n(\zeta)\ \left(1-<\overline{\zeta},z>\right)^2} =$$

$$\begin{aligned}
&= \mathbf{1}_{k_1+k_2\geq N,\, k_1\geq m_1, k_2\geq m_n}\, z_1^{k_1}\, z_2^{k_2}\\
&- \mathbf{1}_{k_1+k_2\geq N}\sum_{p=2}^{n-1} z_1^{m_1}\prod_{j=2,j\neq p}^{n-1}(z_1-\eta_j z_2)^{m_j}\, z_2^{m_n}\sum_{s=0}^{m_p-1} z_2^{m_p-1-s}(z_1-\eta_p z_2)^s\\
&\quad\times\frac{1}{s!}\frac{\partial^s}{\partial t^s}|_{t=\eta_p}\left[\frac{t^{k_1}}{t^{m_1}\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}}\left(\frac{z_2+|\eta_p|^2 z_1/t}{1+|\eta_p|^2}\right)^{k_1+k_2-N+1}\right]\\
&+ \mathbf{1}_{k_1\leq m_1-1, k_2\geq N-k_1}\sum_{p=2}^{n-1} z_1^{m_1}\prod_{j=2,j\neq p}^{n-1}(z_1-\eta_j z_2)^{m_j}\, z_2^{m_n}\sum_{s=0}^{m_p-1} z_2^{m_p-1-s}(z_1-\eta_p z_2)^s\\
&\quad\times\frac{1}{s!}\frac{\partial^s}{\partial t^s}|_{t=\eta_p}\left[\frac{t^{k_1}\, z_2^{k_1+k_2-N+1}}{t^{m_1}\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}}\right]\\
&+ \mathbf{1}_{k_2\leq m_n-1, k_1\geq N-k_2}\sum_{p=2}^{n-1} z_1^{m_1}\prod_{j=2,j\neq p}^{n-1}(z_1-\eta_j z_2)^{m_j}\, z_2^{m_n}\sum_{s=0}^{m_p-1} z_2^{m_p-1-s}(z_1-\eta_p z_2)^s\\
&\quad\times\frac{1}{s!}\frac{\partial^s}{\partial t^s}|_{t=\eta_p}\left[\frac{t^{N-1-k_2}\, z_1^{k_1+k_2-N+1}}{t^{m_1}\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}}\right],
\end{aligned}$$

where

$$\mathbf{1}_{k_1+k_2\geq N} \quad := \quad \begin{cases} 1 & \text{if } k_1+k_2\geq N,\\ 0 & \text{otherwise}\end{cases}$$

(likewise for $\mathbf{1}_{k_1+k_2\geq N,\, k_1\geq m_1, k_2\geq m_n}$, $\mathbf{1}_{k_1\leq m_1-1, k_2\geq N-k_1}$ and $\mathbf{1}_{k_2\leq m_n-1, k_1\geq N-k_2}$).

We begin with the following lemma.

Lemma 5. *For all $r \in [0,1]$ such that $r \neq \alpha_p$, $\forall\, p = 1,\ldots,n$ and all $k_1 \geq 0$, we have*

$$(4.3)\quad \frac{1}{2\pi i}\int_{|\zeta_1|=+\infty}\frac{\zeta_1^{k_1-m_1+1}\ d\zeta_1}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}\ \left(\zeta_1-\frac{r^2 z_1\zeta_2}{\zeta_2-(1-r^2 z_2)}\right)^2} =$$

$$= \mathbf{1}_{k_1\geq m_1+\cdots+m_{n-1}}\ \zeta_2^{k_1-(m_1+\cdots+m_{n-1})}\, P\left(\frac{r^2 z_1}{\zeta_2-(1-r^2)z_2}\right),$$

where $P \in \mathbb{C}[X]$.

Proof. First, notice that, for all $z \in \mathbb{B}_2$ and all $\zeta \in \mathbb{S}_2$, we have by the Cauchy-Schwarz inequality $|1-<\overline{\zeta},z>| \geq 1-\|z\|\,\|\zeta\| = 1-\|z\| > 0$. In addition,

$$(4.4)\qquad |(1-r^2)z_2| = \sqrt{1-r^2}|\,\overline{\zeta}_2 z_2| \leq |\sqrt{1-r^2}\,\|z\| < |\zeta_2|$$

and

$$\left|\frac{r^2 z_1\zeta_2}{\zeta_2-(1-r^2)z_2}\right| = r\frac{|\overline{\zeta_1}z_1|}{|1-\overline{\zeta_2}z_2|} < r = |\zeta_1| \tag{4.5}$$

since $|1-\overline{\zeta}_2 z_2| - |\overline{\zeta}_1 z_1| \geq 1 - |\overline{\zeta}_1 z_1| - |\overline{\zeta}_2 z_2| \geq 1 - \|\zeta\|\,\|z\| > 0$. In particular, $\frac{r^2 z_1\zeta_2}{\zeta_2-(1-r^2)z_2}$ is residue with respect to ζ_1 in the above integral.

Next, by $\int_{|\zeta_1|=+\infty}$ above, we mean $\lim_{R\to+\infty}\int_{|\zeta_1|=R}$ that exists by the residue theorem (it is also $\int_{|\zeta_1|=R}$ for R large enough). If $k_1 < m_1+\cdots+m_{n-1}$ then

$$\deg_{\zeta_1}\left(\frac{\zeta_1^{k_1-m_1+1}}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}\ \left(\zeta_1-\frac{r^2z_1\zeta_2}{\zeta_2-(1-r^2z_2)}\right)^2}\right) \quad\leq\quad -2$$

and

$$\frac{1}{2\pi i}\int_{|\zeta_1|=+\infty}\frac{\zeta_1^{k_1-m_1+1}\,d\zeta_1}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}\ \left(\zeta_1-\frac{r^2z_1\zeta_2}{\zeta_2-(1-r^2z_2)}\right)^2} \quad=\quad 0\,.$$

Now if $k_1 \geq m_1+\cdots+m_{n-1}$, in particular $k_1-m_1+1\geq 0$ and the above integral is

$$\begin{aligned}
&\frac{\partial}{\partial\zeta_1}\Big|_{\zeta_1=\frac{r^2z_1\zeta_2}{\zeta_2-(1-r^2)z_2}}\left[\frac{\zeta_1^{k_1-m_1+1}}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}}\right]\\
&+\sum_{p=2}^{n-1}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial\zeta_1^{m_p-1}}\Big|_{\zeta_1=\eta_p\zeta_2}\left[\frac{\zeta_1^{k-m_1+1}}{\prod_{j=2,j\neq p}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}(\zeta_1-\frac{r^2z_1\zeta_2}{\zeta_2-(1-r^2)z_2})^2}\right]=\\
=\quad&\frac{1}{\zeta_2}\frac{\partial}{\partial x}\Big|_{x=\frac{r^2z_1}{\zeta_2-(1-r^2)z_2}}\left[\frac{(x\zeta_2)^{k_1-m_1+1}}{\prod_{j=2}^{n-1}(x\zeta_2-\eta_j\zeta_2)^{m_j}}\right]\\
&+\sum_{p=2}^{n-1}\frac{1}{(m_p-1)!}\frac{1}{\zeta_2^{m_p-1}}\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}\Big|_{t=\eta_p}\left[\frac{(t\zeta_2)^{k-m_1+1}}{\prod_{j=2,j\neq p}^{n-1}(t\zeta_2-\eta_j\zeta_2)^{m_j}(t\zeta_2-\frac{r^2z_1\zeta_2}{\zeta_2-(1-r^2)z_2})^2}\right]
\end{aligned}$$

$$\begin{aligned}
(4.6)\quad=\quad&\zeta_2^{k_1-(m_1+\cdots+m_{n-1})}\frac{\partial}{\partial x}\Big|_{x=\frac{r^2z_1}{\zeta_2-(1-r^2)z_2}}\left[\frac{x^{k_1-m_1+1}}{\prod_{j=2}^{n-1}(x-\eta_j)^{m_j}}\right]\\
&+\ \zeta_2^{k_1-(m_1+\cdots+m_{n-1})}\sum_{p=2}^{n-1}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}\Big|_{t=\eta_p}\left[\frac{t^{k-m_1+1}}{\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}(t-\frac{r^2z_1}{\zeta_2-(1-r^2)z_2})^2}\right]\\
=\quad&\zeta_2^{k_1-(m_1+\cdots+m_{n-1})}\times
\end{aligned}$$

$$\frac{\partial}{\partial x}\Big|_{x=\frac{r^2z_1}{\zeta_2-(1-r^2)z_2}}\left[\frac{x^{k_1-m_1+1}}{\prod_{j=2}^{n-1}(x-\eta_j)^{m_j}}-\sum_{p=2}^{n-1}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}\Big|_{t=\eta_p}\left(\frac{t^{k-m_1+1}}{\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}(x-t)}\right)\right].$$

Now by lemma 3 and (3.2), we have

$$\frac{X^{k_1-m_1+1}}{\prod_{j=2}^{n-1}(X-\eta_j)^{m_j}} - \sum_{p=2}^{n-1} \frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p}\left(\frac{t^{k-m_1+1}}{\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}(X-t)}\right) =$$

$$= Q\left(X^{k_1-m_1+1}, \prod_{j=2}^{n-1}(X-\eta_j)^{m_j}\right),$$

where Q (resp. R) is the quotient (resp. remainder) of the Euclidean division of $X^{k_1-m_1+1}$ by $\prod_{j=2}^{n-1}(X-\eta_j)^{m_j}$. It follows that this is a polynomial, as well as

$$\frac{\partial Q}{\partial X}|_{X=\frac{r^2 z_1}{\zeta_2-(1-r^2)z_2}}$$

and this proves the lemma.

Notice that it is true as long as $\dfrac{r^2 z_1}{\zeta_2-(1-r^2)z_2} \neq \eta_p$, $\forall\, p=2,\dots,n-1$ then as soon as $|\zeta_2|^2 = 1-r^2 \neq |(1-r^2)z_2 + r^2 z_1/\eta_p|^2$. The lemma is proved for all (r,z) in a dense open set of $[0,1]\times\mathbb{B}_2$ then for all $r\neq\alpha_p$, $p=1,\dots,n$ since the functions that appear in the statement are continuous with respect to r and z.

$\surd$

Now we can give the proof of proposition 3.

Proof. We have to calculate for all k_1, $k_2\geq 0$:

$$(4.7)\qquad -\lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2}\int_{\Sigma_\varepsilon}\frac{\zeta_1^{k_1}\zeta_2^{k_2}\ \omega'\left(\overline{\zeta}\right)\wedge\omega(\zeta)}{g_n(\zeta)\ \left(1-\overline{\zeta_1}z_1-\overline{\zeta_2}z_2\right)^2} =$$

$$= \lim_{\varepsilon\to 0}\sum_{l=1}^{\widetilde{n}-1}\int_{\alpha_{q_l}+\varepsilon}^{\alpha_{q_{l+1}}-\varepsilon} 2rdr\ \times$$

$$\frac{1}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}}\frac{\zeta_2^{k_2-m_n-1}d\zeta_2}{(1-\overline{\zeta_2}z_2)^2}\frac{1}{2\pi i}\int_{|\zeta_1|=r}\frac{\zeta_1^{k_1-m_1-1}d\zeta_1}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}(1-\frac{\overline{\zeta_1}z_1}{1-\overline{\zeta_2}z_2})^2},$$

since

$$\omega'\left(\overline{\zeta}\right)\wedge\omega(\zeta) = -2rdr\wedge\frac{d\zeta_1}{\zeta_1}\wedge\frac{d\zeta_2}{\zeta_2}$$

with the following parametrization of $\zeta\in\mathbb{S}_2$:

$$\begin{cases}\zeta_1 = re^{i\theta_1},\ 0\leq\theta_1<2\pi,\\ \zeta_2=\sqrt{1-r^2}\,e^{i\theta_2},\ 0\leq\theta_2<2\pi,\end{cases}\qquad 0<r<1\,.$$

Now fix $l=1,\dots,\widetilde{n}-1$ and $\alpha_{q_l}<r<\alpha_{q_{l+1}}$. Then for all $|\zeta_1|=r$ and all $|\zeta_2|=\sqrt{1-r^2}$, one has $|\eta_{q_l}|^2<r^2(1+|\eta_{q_l}|^2)$ thus $|\eta_{q_l}|\sqrt{1-r^2}<r$ (similarly $|\eta_{q_{l+1}}|\sqrt{1-r^2}>r$). It follows that

$$(4.8)\qquad \begin{cases}|\eta_2\zeta_2|\leq\cdots\leq|\eta_{q_l}\zeta_2|<|\zeta_1|\,,\\ |\zeta_1|<|\eta_{q_l+1}\zeta_2|=\cdots=|\eta_{q_{l+1}}\zeta_2|\leq\cdots\leq|\eta_{n-1}\zeta_2|\,.\end{cases}$$

This yields to

$$\frac{1}{2\pi i}\int_{|\zeta_1|=r}\frac{\zeta_1^{k_1-m_1+1}\,d\zeta_1}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}(\zeta_1-\frac{r^2z_1\zeta_2}{\zeta_2-(1-r^2)z_2})^2} \quad =$$

$$\begin{aligned}
&= \frac{1}{2\pi i}\int_{|\zeta_1|=+\infty}\frac{\zeta_1^{k_1-m_1+1}\,d\zeta_1}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}(\zeta_1-\frac{r^2z_1\zeta_2}{\zeta_2-(1-r^2)z_2})^2}\\
&\quad -\sum_{p=q_l+1}^{n-1}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial\zeta_1^{m_p-1}}|_{\zeta_1=\eta_p\zeta_2}\left(\frac{\zeta_1^{k_1-m_1+1}}{\prod_{j=2,j\neq p}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}(\zeta_1-\frac{r^2z_1\zeta_2}{\zeta_2-(1-r^2)z_2})^2}\right)\\
&= \mathbf{1}_{k_1\geq m_1+\cdots+m_{n-1}}\,\zeta_2^{k_1-(m_1+\cdots+m_{n-1})}P\left(\frac{r^2z_1}{\zeta_2-(1-r^2)z_2}\right)\\
&\quad -\zeta_2^{k_1-(m_1+\cdots+m_{n-1})}\sum_{p=q_l+1}^{n-1}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p}\left(\frac{t^{k_1-m_1+1}}{\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}(t-\frac{r^2z_1}{\zeta_2-(1-r^2)z_2})^2}\right)
\end{aligned}$$

by lemma 5.

One can deduce that

$$\frac{1}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}}\frac{\zeta_2^{k_2-m_n+1}d\zeta_2}{(\zeta_2-(1-r^2)z_2)^2}\frac{1}{2\pi i}\int_{|\zeta_1|=r}\frac{\zeta_1^{k_1-m_1+1}d\zeta_1}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}(\zeta_1-\frac{r^2z_1\zeta_2}{\zeta_2-(1-r^2)z_2})^2} \quad =$$

$$(4.9)\qquad = \mathbf{1}_{k_1\geq m_1+\cdots+m_{n-1}}\frac{1}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}}\frac{\zeta_2^{k_1+k_2-N+1}P(\frac{r^2z_1}{\zeta_2-(1-r^2)z_2})}{(\zeta_2-(1-r^2)z_2)^2}\,d\zeta_2$$

$$-\sum_{p=q_l+1}^{n-1}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p}\left[\frac{t^{k_1-m_1+1}}{\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}}\frac{1}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}}\frac{\zeta_2^{k_1+k_2-N+1}\,d\zeta_2}{(t(\zeta_2-(1-r^2)z_2)-r^2z_1)^2}\right].$$

Now we see in the first integral that the only possible residues are $\zeta_2=(1-r^2)z_2$ and $\zeta_2=0$. If $k_1+k_2<N$ (and $P\neq 0$ otherwise the integral is zero) then

$$\deg\left(\frac{\zeta_2^{k_1+k_2-N+1}}{(\zeta_2-(1-r^2)z_2)^2}P\left(\frac{r^2z_1}{\zeta_2-(1-r^2)z_2}\right)\right) \quad\leq\quad -2$$

and

$$\frac{1}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}}\frac{\zeta_2^{k_1+k_2-N+1}P(\frac{r^2z_1}{\zeta_2-(1-r^2)z_2})d\zeta_2}{(\zeta_2-(1-r^2)z_2)^2}=\frac{1}{2\pi i}\int_{|\zeta_2|=+\infty}\frac{\zeta_2^{k_1+k_2-N+1}P(\frac{r^2z_1}{\zeta_2-(1-r^2)z_2})d\zeta_2}{(\zeta_2-(1-r^2)z_2)^2}=0\,.$$

If $k_1+k_2\geq N$ then the only residue is $\zeta_2=(1-r^2)z_2$ and we get by (4.6)

$$\frac{1}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}}\frac{\zeta_2^{k_1+k_2-N+1}P(\frac{r^2z_1}{\zeta_2-(1-r^2)z_2})}{(\zeta_2-(1-r^2)z_2)^2}\,d\zeta_2 \quad =$$

$$= \lim_{\varepsilon'\to 0} \frac{1}{2\pi i} \int_{|\zeta_2-(1-r^2)z_2|=\varepsilon'} \frac{\zeta_2^{k_1+k_2-N+1} P(\frac{r^2 z_1}{\zeta_2-(1-r^2)z_2})}{(\zeta_2-(1-r^2)z_2)^2}\, d\zeta_2\,,$$

$$= \lim_{\varepsilon'\to 0} \frac{1}{2\pi i} \int_{|\zeta_2-(1-r^2)z_2|=\varepsilon'} \frac{\zeta_2^{k_1+k_2-N+1}}{(\zeta_2-(1-r^2)z_2)^2} \frac{\partial}{\partial x}\Big|_{x=\frac{r^2 z_1}{\zeta_2-(1-r^2)z_2}} \left[\frac{x^{k_1-m_1+1}}{\prod_{j=2}^{n-1}(x-\eta_j)^{m_j}}\right] d\zeta_2$$

$$+ \lim_{\varepsilon'\to 0} \frac{1}{2\pi i} \int_{|\zeta_2-(1-r^2)z_2|=\varepsilon'} \frac{\zeta_2^{k_1+k_2-N+1}}{(\zeta_2-(1-r^2)z_2)^2}\, d\zeta_2 \;\times$$

$$\times \sum_{p=2}^{n-1} \frac{1}{(m_p-1)!} \frac{\partial^{m_p-1}}{\partial t^{m_p-1}}\Big|_{t=\eta_p} \left[\frac{t^{k_1-m_1+1}}{\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}(t-\frac{r^2 z_1}{\zeta_2-(1-r^2)z_2})^2}\right]$$

$$= \lim_{\varepsilon'\to 0} (r^2 z_1)^{k_1-m_1}\, \frac{\partial}{\partial t}\Big|_{t=1}\, t^{k_1-m_1+1} \;\times$$

$$\frac{1}{2\pi i} \int_{|\zeta_2-(1-r^2)z_2|=\varepsilon'} \frac{\zeta_2^{k_1+k_2-N+1}\, d\zeta_2}{(\zeta_2-(1-r^2)z_2)^{k_1-(m_1+\cdots+m_{n-1})+2} \prod_{j=2}^{n-1}(tr^2 z_1-\eta_j(\zeta_2-(1-r^2)z_2))^{m_j}}$$

$$+ \lim_{\varepsilon'\to 0} \sum_{p=2}^{n-1} \frac{1}{(m_p-1)!} \;\times$$

$$\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}\Big|_{t=\eta_p} \left[\frac{t^{k_1-m_1+1}}{\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}} \frac{1}{2\pi i} \int_{|\zeta_2-(1-r^2)z_2|=\varepsilon'} \frac{\zeta_2^{k_1+k_2-N+1}\, d\zeta_2}{(t(\zeta_2-(1-r^2)z_2)-r^2 z_1)^2}\right]$$

(one can switch integral and derivative for all fixed z and all ε' small enough since the above functions, as well as all their derivatives with respect to t, are integrable).

Now z_1, r, $t \neq 0$ being fixed (since $z \in U_\eta$), one can choose for ζ_2 a small enough neighborhood of $(1-r^2)z_2$ such that the function

$$\zeta_2 \quad\mapsto\quad \frac{\zeta_2^{k_1+k_2-N+1}}{(t(\zeta_2-(1-r^2)z_2)-r^2 z_1)^2}$$

has no singularity. It follows that

$$\frac{1}{2\pi i} \int_{|\zeta_2|=\sqrt{1-r^2}} \frac{\zeta_2^{k_1+k_2-N+1} P(\frac{r^2 z_1}{\zeta_2-(1-r^2)z_2})}{(\zeta_2-(1-r^2)z_2)^2}\, d\zeta_2 \quad =$$

$$= \quad (r^2 z_1)^{k_1-m_1}\, \frac{\partial}{\partial t}\Big|_{t=1}\, t^{k_1-m_1+1} \frac{1}{(k_1-(m_1+\cdots+m_{n-1})+1)!} \;\times$$

$$\times \frac{\partial^{k_1-(m_1+\cdots+m_{n-1})+1}}{\partial \zeta_2^{k_1-(m_1+\cdots+m_{n-1})+1}}\Big|_{\zeta_2=(1-r^2)z_2} \left[\frac{\zeta_2^{k_1+k_2-N+1}}{\prod_{j=2}^{n-1}(tr^2 z_1-\eta_j(\zeta_2-(1-r^2)z_2))^{m_j}}\right]$$

$$
\begin{aligned}
&= (r^2z_1)^{k_1-m_1}\ \frac{\partial}{\partial t}|_{t=1}\ t^{k_1-m_1+1}\ \times \\
&\quad \times \sum_{v_1+\cdots+v_{n-1}=k_1-(m_1+\cdots+m_{n-1})+1,\, v_1\le k_1+k_2-N+1} \frac{(k_1+k_2-N+1)!}{v_1!\,(k_1+k_2-N+1-v_1)!}\ \times \\
&\quad \times ((1-r^2)z_2)^{k_1+k_2-N+1-v_1} \prod_{j=2}^{n-1} \frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\ \frac{\eta_j^{v_j}}{(tr^2z_1)^{v_j+m_j}} \\
&= \sum_{v_1+\cdots+v_{n-1}=k_1-(m_1+\cdots+m_{n-1})+1,\, v_1\le k_1+k_2-N+1} \frac{(k_1+k_2-N+1)!}{v_1!\,(k_1+k_2-N+1-v_1)!}\ \times \\
&\quad \times (r^2z_1)^{k_1-m_1-(v_2+m_2+\cdots+v_{n-1}+m_{n-1})}((1-r^2)z_2)^{k_1+k_2-N+1-v_1}\ \times \\
&\quad \times \prod_{j=2}^{n-1} \frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\ \eta_j^{v_j}\ \frac{d}{dt}|_{t=1}\left(t^{k_1-m_1+1-(v_2+m_2+\cdots+v_{n-1}+m_{n-1})}\right) \\
&= \sum_{v_1+\cdots+v_{n-1}=k_1-(m_1+\cdots+m_{n-1})+1,\, v_1\le k_1+k_2-N+1,\, v_1\ge 1} \frac{(k_1+k_2-N+1)!}{v_1!\,(k_1+k_2-N+1-v_1)!}\ \times \\
&\quad \times (r^2z_1)^{v_1-1}((1-r^2)z_2)^{k_1+k_2-N+1-v_1} \prod_{j=2}^{n-1} \frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\ \eta_j^{v_j}\ v_1
\end{aligned}
$$

$$
\begin{aligned}
(4.10) &= \sum_{v_1+\cdots+v_{n-1}=k_1-(m_1+\cdots+m_{n-1}),\, v_1\le k_1+k_2-N} \frac{(k_1+k_2-N+1)!}{v_1!\,(k_1+k_2-N-v_1)!}\ \times \\
&\quad \times (r^2z_1)^{v_1}((1-r^2)z_2)^{k_1+k_2-N-v_1} \prod_{j=2}^{n-1} \frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\ \eta_j^{v_j}\,.
\end{aligned}
$$

Now consider the other part of (4.9). In order to calculate the following integral, for all t close to η_p, $p = q_l+1,\ldots,n-1$,

$$\frac{1}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}} \frac{\zeta_2^{k_1+k_2-N+1}\, d\zeta_2}{(\zeta_2-(1-r^2)z_2-r^2z_1/t)^2}\ ,$$

the following lemma will be useful.

Lemma 6. *Let $K\subset\mathbb{B}_2$ be a compact subset. There exists $\varepsilon_K>0$ such that, for all $z\in K$, for all $p=q_l+1,\ldots,n-1$, for all t close to η_p and all $r<\alpha_{q_{l+1}}$,*

$$\left|\left(1-r^2\right)z_2+\frac{r^2z_1}{t}\right| \quad\le\quad \left(1-\varepsilon_K^2\right)\sqrt{1-r^2}\,. \tag{4.11}$$

In particular, for all $z\in\mathbb{B}_2$, for all t close to η_p and all $r<\alpha_{q_{l+1}}$, one has

$$\left|\left(1-r^2\right)z_2+\frac{r^2z_1}{t}\right| \quad<\quad \sqrt{1-r^2}\,.$$

These assertions are still true for all $p=2,\ldots,n-1$, for all t close enough to η_p and r to α_p.

Proof. One has by the Cauchy-Schwarz inequality

$$|(1-r^2)z_2+r^2z_1/t| \quad\leq\quad \|z\|\sqrt{(1-r^2)^2+\frac{r^4}{|t|^2}} \;\leq\; \|z\|\sqrt{1-2r^2+r^4\left(1+\frac{1}{|t|^2}\right)}\,.$$

Since $z\in K\subset\mathbb{B}_2$, there is $\varepsilon_K>0$ such that $\|z\|\leq 1-\varepsilon_K$. It follows that, for all t close enough to η_p,

$$\begin{aligned}|(1-r^2)z_2+r^2z_1/t| &\leq (1-\varepsilon_K)\,(1+\varepsilon_K)\sqrt{1-2r^2+\frac{r^4}{\alpha_p^2}}\\ &\leq (1-\varepsilon_K^2)\sqrt{1-2r^2+\frac{r^4}{\alpha_{q_{l+1}}^2}}\\ &< \sqrt{1-r^2}\,.\end{aligned}$$

Similarly, since

$$\lim_{t\to\eta_p,r\to\alpha_p}\sqrt{1-2r^2+r^4\left(1+\frac{1}{|t|^2}\right)} \;=\; \sqrt{1-\alpha_p^2} \;=\; \lim_{r\to\alpha_p}\sqrt{1-r^2}\,,$$

one has, for all t close enough to η_p and r to α_p,

$$\begin{aligned}|(1-r^2)z_2+r^2z_1/t| &\leq (1-\varepsilon_K)\sqrt{1-2r^2+r^4\left(1+\frac{1}{|t|^2}\right)}\\ &\leq (1-\varepsilon_K)^2\sqrt{1-r^2}\,.\end{aligned}$$

√

In particular, $\zeta_2=(1-r^2)z_2+r^2z_1/t$ is residue in the above integral. If $k_1+k_2<N$ then

$$\frac{1}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}}\frac{\zeta_2^{k_1+k_2-N+1}\,d\zeta_2}{(\zeta_2-(1-r^2)z_2-r^2z_1/t)^2}=\frac{1}{2\pi i}\int_{|\zeta_2|=+\infty}\frac{\zeta_2^{k_1+k_2-N+1}\,d\zeta_2}{(\zeta_2-(1-r^2)z_2-r^2z_1/t)^2}=0\,.$$

If $k_1+k_2\geq N$ one has

$$\frac{1}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}}\frac{\zeta_2^{k_1+k_2-N+1}\,d\zeta_2}{(\zeta_2-(1-r^2)z_2-r^2z_1/t)^2}=(k_1+k_2-N+1)((1-r^2)z_2+r^2z_1/t)^{k_1+k_2-N}\,.$$

It follows by (4.9) and (4.10) that, for all k_1, $k_2\geq 0$,

$$\frac{1}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}}\frac{\zeta_2^{k_2-m_n+1}d\zeta_2}{(\zeta_2-(1-r^2)z_2)^2}\frac{1}{2\pi i}\int_{|\zeta_1|=r}\frac{\zeta_1^{k_1-m_1+1}d\zeta_1}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}(\zeta_1-\frac{r^2z_1\zeta_2}{\zeta_2-(1-r^2)z_2})^2} \quad =$$

$$= \mathbf{1}_{k_1+k_2\geq N} \sum_{v_1+\cdots+v_{n-1}=k_1-(m_1+\cdots+m_{n-1}),\, v_1\leq k_1+k_2-N} \frac{(k_1+k_2-N+1)!}{v_1!\,(k_1+k_2-N-v_1)!} \times$$

$$\times\, (r^2 z_1)^{v_1} ((1-r^2)z_2)^{k_1+k_2-N-v_1} \prod_{j=2}^{n-1} \frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\, \eta_j^{v_j}$$

$$- \mathbf{1}_{k_1+k_2\geq N} \sum_{p=q_l+1}^{n-1} \frac{1}{(m_p-1)!} \times$$

$$\times\, \frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p} \left(\frac{(k_1+k_2-N+1)t^{k_1-m_1-1}((1-r^2)z_2+r^2z_1/t)^{k_1+k_2-N}}{\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}} \right)$$

(notice that the condition $\mathbf{1}_{k_1\geq m_1+\cdots+m_{n-1}}$ in (4.9) is satisfied since if $k_1 < m_1 + \cdots + m_{n-1}$ then $\sum_{v_1+\cdots+v_{n-1}=k_1-(m_1+\cdots+m_{n-1})} = 0$).

This is valid for all $l = 1, \ldots, \widetilde{n}-1$ and all $\alpha_{q_l} < r < \alpha_{q_{l+1}}$. It follows from (4.7) that, for all $k_1+k_2 \geq N$ (that we will assume in the following since for $k_1+k_2 < N$ the propostition is proved),

$$- \lim_{\varepsilon\to 0} \frac{1}{(2\pi i)^2} \int_{\Sigma_\varepsilon} \frac{\zeta_1^{k_1}\zeta_2^{k_2}\ \omega'\left(\overline{\zeta}\right)\wedge\omega(\zeta)}{g_n(\zeta)\ \left(1-\overline{\zeta_1}z_1-\overline{\zeta_2}z_2\right)^2} =$$

$$= \sum_{v_1+\cdots+v_{n-1}=k_1-(m_1+\cdots+m_{n-1}),\, v_1\leq k_1+k_2-N} \frac{(k_1+k_2-N+1)!}{v_1!\,(k_1+k_2-N-v_1)!} z_1^{v_1} z_2^{k_1+k_2-N-v_1}$$

$$\times \prod_{j=2}^{n-1} \frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\, \eta_j^{v_j} \lim_{\varepsilon\to 0} \sum_{l=1}^{\widetilde{n}-1} \int_{\alpha_{q_l}+\varepsilon}^{\alpha_{q_{l+1}}-\varepsilon} (r^2)^{v_1}(1-r^2)^{k_1+k_2-N-v_1}\ 2rdr$$

$$- \lim_{\varepsilon\to 0} \sum_{l=1}^{\widetilde{n}-1} \sum_{p=q_l+1}^{n-1} \int_{\alpha_{q_l}+\varepsilon}^{\alpha_{q_{l+1}}-\varepsilon} 2rdr \times$$

$$\frac{1}{(m_p-1)!} \frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p} \left(\frac{(k_1+k_2-N+1)t^{k_1-m_1-1}((1-r^2)z_2+r^2z_1/t)^{k_1+k_2-N}}{\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}} \right).$$

Now notice that, in the last above sum, by (2.13) we have $p \geq q_l+1$ if and only if $\alpha_p > \alpha_{q_l}$ that is equivalent to $\alpha_p \geq \alpha_{q_{l+1}}$, so $l = 1, \ldots, l_p-1$ where l_p is defined such that $\alpha_{q_{l_p}} = \alpha_p$. This allows us to get

$$\sum_{v_1+\cdots+v_{n-1}=k_1-(m_1+\cdots+m_{n-1}), v_1\leq k_1+k_2-N} z_1^{v_1} z_2^{k_1+k_2-N-v_1} \prod_{j=2}^{n-1} \frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\, \eta_j^{v_j} \times$$

$$\times \frac{(k_1+k_2-N+1)!}{v_1!\,(k_1+k_2-N-v_1)!} \int_0^1 x^{v_1}(1-x)^{k_1+k_2-N-v_1}\ dx$$

$$- \sum_{p=2}^{n-1} \int_0^{\alpha_p} 2rdr\ \frac{1}{(m_p-1)!} \times$$

$$\times\, \frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p} \left(\frac{(k_1+k_2-N+1)t^{k_1-m_1-1}((1-r^2)z_2+r^2z_1/t)^{k_1+k_2-N}}{\prod_{j=2,j\neq p}^{n-1}\ (t-\eta_j)^{m_j}} \right)$$

$$(4.12) \quad = \sum_{v_1+\cdots+v_{n-1}=k_1-(m_1+\cdots+m_{n-1}),\, v_1\le k_1+k_2-N} z_1^{v_1}\, z_2^{k_1+k_2-N-v_1} \times$$
$$\times \prod_{j=2}^{n-1} \frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\, \eta_j^{v_j}$$

$$- \sum_{p=2}^{n-1} \frac{1}{(m_p-1)!} \frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p} \left[\frac{t^{k_1-m_1-1}\left((z_2+\alpha_p^2(z_1/t-z_2))^{k_1+k_2-N+1} - z_2^{k_1+k_2-N+1}\right)}{\prod_{j=2,j\neq p}^{n-1} (t-\eta_j)^{m_j}\ \ (z_1/t-z_2)} \right],$$

the last equality coming from the following idendity (that can be proved by induction on $v_1 = 0, \ldots, k_1+k_2-N+1$ with integrating by parts)

$$\int_0^1 x^{v_1}(1-x)^{k_1+k_2-N-v_1}\, dx \quad = \quad \frac{v_1!\,(k_1+k_2-N-v_1)!}{(k_1+k_2-N+1)!}$$

(on the other hand, notice that, since $z \in U_\eta$, then for all $p = 2, \ldots, n-1$ and all t close enough to η_p, $z_1/t - z_2 \neq 0$).

Now assume that $k_1 \le m_1 - 1$. Then $k_1 - (m_1+\cdots+m_{n-1}) < 0$ and $k_2 \ge N - k_1 \ge 0$. We get from above

$$- \lim_{\varepsilon\to 0} \frac{1}{(2\pi i)^2} \int_{\Sigma_\varepsilon} \frac{\zeta_1^{k_1}\zeta_2^{k_2}\ \omega'\left(\overline{\zeta}\right)\wedge\omega(\zeta)}{g_n(\zeta)\ \left(1-<\overline{\zeta},z>\right)^2} \quad =$$

$$(4.13) \ = \ \sum_{p=2}^{n-1} \frac{1}{(m_p-1)!} \frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p} \left[\frac{t^{k_1}\ z_2^{k_1+k_2-N+1}}{(z_1-tz_2)\ t^{m_1}\ \prod_{j=2,j\neq p}^{n-1} (t-\eta_j)^{m_j}} \right]$$

$$- \sum_{p=2}^{n-1} \frac{1}{(m_p-1)!} \times$$
$$\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p} \left[\frac{t^{k_1}}{(z_1-tz_2)\ t^{m_1}\ \prod_{j=2,j\neq p}^{n-1} (t-\eta_j)^{m_j}} \left(\frac{z_2+|\eta_p|^2 z_1/t}{1+|\eta_p|^2} \right)^{k_1+k_2-N+1} \right].$$

Else, we have $k_1 \ge m_1$. Now assume that $k_2 \ge m_n$ then $k_1+k_2-N \ge k_1-(m_1+\cdots+m_{n-1})$ and we get

$$\sum_{v_1+\cdots+v_{n-1}=k_1-(m_1+\cdots+m_{n-1})} z_1^{v_1}\, z_2^{k_1+k_2-N-v_1} \prod_{j=2}^{n-1} \frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\, \eta_j^{v_j} \quad =$$

$$= \quad z_2^{k_1+k_2-N} \sum_{u=0}^{k_1-m_1-(m_2+\cdots+m_{n-1})} (z_1/z_2)^{k_1-m_1-(m_2+\cdots+m_{n-1})-u} \times$$

$$\times \sum_{v_2+\cdots+v_{n-1}=u} \prod_{j=2}^{n-1} \frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\,\eta_j^{v_j}$$

$$= \quad z_2^{k_1+k_2-N}\, Q\left(X^{k_1-m_1}, \prod_{j=2}^{n-1}(X-\eta_j)^{m_j}\right)|_{X\,=\,z_1/z_2}$$

by lemma 3. On the other hand, we have

$$\sum_{p=2}^{n-1} \frac{1}{(m_p-1)!}\,\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p} \left[\frac{t^{k_1-m_1}}{(z_1/z_2-t)\ \prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}}\right] \quad =$$

$$= \quad \left[\frac{R\left(X^{k_1-m_1}, \prod_{j=2}^{n-1}(X-\eta_j)^{m_j}\right)}{\prod_{j=2}^{n-1}(X-\eta_j)^{m_j}}\right]|_{X\,=\,z_1/z_2}.$$

It follows by (4.12) that

$$-\lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2}\int_{\Sigma_\varepsilon}\frac{\zeta_1^{k_1}\zeta_2^{k_2}\ \omega'\left(\overline{\zeta}\right)\wedge\omega(\zeta)}{g_n(\zeta)\ \left(1-<\overline{\zeta},z>\right)^2} \quad = \quad z_2^{k_1+k_2-N}\frac{(z_1/z_2)^{k_1-m_1}}{\prod_{j=2}^{n-1}(z_1/z_2-\eta_j)^{m_j}}$$

$$-\sum_{p=2}^{n-1}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p}\left[\frac{t^{k_1-m_1}(z_2+\alpha_p^2(z_1/t-z_2))^{k_1+k_2-N+1}}{(z_1-tz_2)\ \prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}}\right]$$

$$(4.14)\quad = \frac{z_1^{k_1}z_2^{k_2}}{g_n(z)} - \sum_{p=2}^{n-1}\frac{1}{(m_p-1)!} \times$$

$$\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p}\left[\frac{t^{k_1-m_1}}{(z_1-tz_2)\ \prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}}\left(\frac{z_2+|\eta_p|^2z_1/t}{1+|\eta_p|^2}\right)^{k_1+k_2-N+1}\right].$$

Lastly, we have $k_2 \leq m_n-1$ then $k_1 \geq N-k_2 \geq m_1$ and $k_1+k_2-N \leq k_1-(m_1+\cdots+m_{n-1})-1$. It follows that

$$\sum_{v_1+\cdots+v_{n-1}=k_1-(m_1+\cdots+m_{n-1}),\, v_1\leq k_1+k_2-N} z_1^{v_1}\ z_2^{k_1+k_2-N-v_1}\ \prod_{j=2}^{n-1}\frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\,\eta_j^{v_j} \ =$$

$$= \quad z_2^{k_1+k_2-N}\, Q\left(X^{k_1-m_1}, \prod_{j=2}^{n-1}(X-\eta_j)^{m_j}\right)|_{X\,=\,z_1/z_2}$$

$$-\sum_{v_1+\cdots+v_{n-1}=k_1-(m_1+\cdots+m_{n-1}),\, v_1\geq k_1+k_2-N+1} z_1^{v_1}\ z_2^{k_1+k_2-N-v_1}\ \prod_{j=2}^{n-1}\frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\,\eta_j^{v_j}$$

$$= \quad z_2^{k_1+k_2-N}\, Q\left(X^{k_1-m_1}, \prod_{j=2}^{n-1}(X-\eta_j)^{m_j}\right)|_{X\,=\,z_1/z_2}$$

$$- \frac{z_1^{k_1+k_2-N+1}}{z_2} \sum_{v_1+\cdots+v_{n-1}=m_n-1-k_2} (z_1/z_2)^{v_1} \prod_{j=2}^{n-1} \frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\,\eta_j^{v_j}\,.$$

Notice by lemma 3 (since $N-m_1 \geq m_n \geq k_2+1$) that

$$\sum_{u=0}^{m_n-1-k_2} (z_1/z_2)^{m_n-1-k_2-u} \sum_{v_2+\cdots+v_{n-1}=u} \prod_{j=2}^{n-1} \frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\,\eta_j^{v_j} \quad =$$

$$= \quad Q\left(X^{N-m_1-1-k_2}, \prod_{j=2}^{n-1}(X-\eta_j)^{m_j}\right)|_{X\,=\,z_1/z_2}$$

$$= \quad \frac{(z_1/z_2)^{N-m_1-1-k_2}}{\prod_{j=2}^{n-1}(z_1/z_2-\eta_j)^{m_j}} - \sum_{p=2}^{n-1} \frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p}\left[\frac{t^{N-m_1-1-k_2}}{(z_1/z_2-t)\;\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}}\right].$$

It follows from (4.12) that

$$-\lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2}\int_{\Sigma_\varepsilon}\frac{\zeta_1^{k_1}\zeta_2^{k_2}\;\omega'\left(\overline{\zeta}\right)\wedge\omega(\zeta)}{g_n(\zeta)\;\left(1-<\overline{\zeta},z>\right)^2} \quad =$$

$$= \quad z_2^{k_1+k_2-N}\frac{(z_1/z_2)^{k_1-m_1}}{\prod_{j=2}^{n-1}(z_1/z_2-\eta_j)^{m_j}} - \frac{z_1^{k_1+k_2-N+1}}{z_2}\frac{(z_1/z_2)^{N-m_1-1-k_2}}{\prod_{j=2}^{n-1}(z_1/z_2-\eta_j)^{m_j}}$$

$$+ \quad \frac{z_1^{k_1+k_2-N+1}}{z_2}\sum_{p=2}^{n-1}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p}\left[\frac{t^{N-m_1-1-k_2}}{(z_1/z_2-t)\;\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}}\right]$$

$$- \quad \sum_{p=2}^{n-1}\frac{1}{(m_p-1)!} \times$$

$$\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p}\left[\frac{t^{k_1-m_1}}{(z_1-tz_2)\;\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}}\left(\frac{z_2+|\eta_p|^2 z_1/t}{1+|\eta_p|^2}\right)^{k_1+k_2-N+1}\right]$$

$$(4.15) \quad = \quad \sum_{p=2}^{n-1}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p}\left[\frac{t^{N-m_1-1-k_2}\;z_1^{k_1+k_2-N+1}}{(z_1-tz_2)\;\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}}\right]$$

$$- \sum_{p=2}^{n-1}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p}\left[\frac{t^{k_1-m_1}}{(z_1-tz_2)\;\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}}\left(\frac{z_2+|\eta_p|^2 z_1/t}{1+|\eta_p|^2}\right)^{k_1+k_2-N+1}\right].$$

Finally, we get from (4.12), (4.13), (4.14) and (4.15)

$$-\lim_{\varepsilon\to 0}\frac{g_n(z)}{(2\pi i)^2}\int_{\Sigma_\varepsilon}\frac{\zeta_1^{k_1}\zeta_2^{k_2}\;\omega'\left(\overline{\zeta}\right)\wedge\omega(\zeta)}{g_n(\zeta)\;\left(1-<\overline{\zeta},z>\right)^2} \quad =$$

$$
\begin{aligned}
= \quad & \mathbf{1}_{k_1+k_2\geq N, k_1\geq m_1, k_2\geq m_n}\, z_1^{k_1}\, z_2^{k_2} \\
& - \mathbf{1}_{k_1+k_2\geq N} \sum_{p=2}^{n-1} z_1^{m_1} \prod_{j=2, j\neq p}^{n-1} (z_1-\eta_j z_2)^{m_j}\, z_2^{m_n} \; \times \\
& \times \frac{(z_1-\eta_p z_2)^{m_p}}{(m_p-1)!} \frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p} \frac{t^{k_1} \left(\frac{z_2+|\eta_p|^2 z_1/t}{1+|\eta_p|^2}\right)^{k_1+k_2-N+1}}{(z_1-tz_2)\, t^{m_1} \prod_{j=2, j\neq p}^{n-1} (t-\eta_j)^{m_j}} \\
& + \mathbf{1}_{k_1\leq m_1-1, k_2\geq N-k_1} \sum_{p=2}^{n-1} z_1^{m_1} \prod_{j=2, j\neq p}^{n-1} (z_1-\eta_j z_2)^{m_j}\, z_2^{m_n} \; \times \\
& \times \frac{(z_1-\eta_p z_2)^{m_p}}{(m_p-1)!} \frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p} \frac{t^{k_1}\, z_2^{k_1+k_2-N+1}}{(z_1-tz_2)\, t^{m_1} \prod_{j=2, j\neq p}^{n-1} (t-\eta_j)^{m_j}} \\
& + \mathbf{1}_{k_2\leq m_n-1, k_1\geq N-k_2} \sum_{p=2}^{n-1} z_1^{m_1} \prod_{j=2, j\neq p}^{n-1} (z_1-\eta_j z_2)^{m_j}\, z_2^{m_n} \; \times \\
& \times \frac{(z_1-\eta_p z_2)^{m_p}}{(m_p-1)!} \frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p} \frac{t^{N-1-k_2}\, z_1^{k_1+k_2-N+1}}{(z_1-tz_2)\, t^{m_1} \prod_{j=2, j\neq p}^{n-1} (t-\eta_j)^{m_j}}
\end{aligned}
$$

and the proof of the proposition is achieved.

$\surd$

5. Calculation of the interpolation part

In this section, we will prove the following result that allows us to complete the proof of theorem 1. We assume that there exists $m_p \geq 1,\ 2 \leq p \leq n-1$.

Proposition 4. *Let be $f \in \mathcal{O}(\mathbb{B}_2)$ and $f(z) = \sum_{k_1,k_2\geq 0} a_{k_1,k_2} z_1^{k_1} z_2^{k_2}$ its Taylor expansion. For all $z \in \mathbb{B}_2$,*

$$
\lim_{\varepsilon\to 0} \frac{1}{(2\pi i)^2} \int_{\partial\widetilde{\Sigma}_\varepsilon} \frac{\zeta_1^{k_1}\zeta_2^{k_2} \det\left(\overline{\zeta}, P_n(\zeta,z)\right)}{g_n(\zeta)\left(1-<\overline{\zeta}, z>\right)}\, \omega(\zeta) \; = \tag{5.1}
$$

$$
\begin{aligned}
= \; & \sum_{u_1=0}^{m_1-1} z_1^{u_1} \prod_{j=2}^{n-1} (z_1-\eta_j z_2)^{m_j} z_2^{m_n} \sum_{p=2}^{n-1} \frac{1}{(m_p-1)!} \frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p} \; \frac{1}{t^{u_1+1} \prod_{j=2, j\neq p}^{n-1} (t-\eta_j)^{m_j}} \\
& \times \{\, \mathbf{1}_{k_1+k_2\geq N_{u_1}} \frac{1}{1+|\eta_p|^2} t^{k_1} \left(\frac{z_2+|\eta_p|^2 z_1/t}{1+|\eta_p|^2}\right)^{k_1+k_2-N_{u_1}} \\
& - \; \mathbf{1}_{k_1\leq u_1,\, k_2\geq N_{u_1}-k_1}\, t^{k_1} z_2^{k_1+k_2-N_{u_1}} \;\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{p=2}^{n-1} \sum_{u_p=0}^{m_p-1} (z_1-\eta_p z_2)^{u_p} \prod_{j=p+1}^{n-1} (z_1-\eta_j z_2)^{m_j} z_2^{m_n} \times \\
&\times \; \{ \mathbf{1}_{k_1+k_2\geq N_{u_p}} \{ \frac{1}{u_p!} \frac{\partial^{u_p}}{\partial t^{u_p}}|_{t=\eta_p} \frac{1+|\eta_p|^2\eta_p/t}{1+|\eta_p|^2} \frac{t^{k_1}\left(\frac{z_2+|\eta_p|^2 z_1/t}{1+|\eta_p|^2}\right)^{k_1+k_2-N_{u_p}}}{\prod_{j=p+1}^{n-1}(t-\eta_j)^{m_j}} \\
&+ \sum_{q=p+1}^{n-1} \frac{1}{(m_q-1)!} \frac{\partial^{m_q-1}}{\partial t^{m_q-1}}|_{t=\eta_q} \frac{1+|\eta_q|^2\eta_p/t}{1+|\eta_q|^2} \frac{t^{k_1}\left(\frac{z_2+|\eta_q|^2 z_1/t}{1+|\eta_q|^2}\right)^{k_1+k_2-N_{u_p}}}{(t-\eta_p)^{u_p+1}\prod_{j=p+1,j\neq q}^{n-1}(t-\eta_j)^{m_j}} \; \} \\
&- \; \mathbf{1}_{k_2\leq m_n-1, k_1\geq N_{u_p}-k_2}\, \eta_p\, z_1^{k_1+k_2-N_{u_p}} \times \{ \frac{1}{u_p!} \frac{\partial^{u_p}}{\partial t^{u_p}}|_{t=\eta_p} \frac{t^{N_{u_p}-1-k_2}}{\prod_{j=p+1}^{n-1}(t-\eta_j)^{m_j}} \\
&\quad + \sum_{q=p+1}^{n-1} \frac{1}{(m_q-1)!} \frac{\partial^{m_q-1}}{\partial t^{m_q-1}}|_{t=\eta_q} \frac{t^{N_{u_p}-1-k_2}}{(t-\eta_p)^{u_p+1}\prod_{j=p+1,j\neq q}^{n-1}(t-\eta_j)^{m_j}} \; \} \\
&+ \; \mathbf{1}_{k_1\geq 0, k_2\leq m_n-1}\, z_1^{k_1} z_2^{k_2} \; .
\end{aligned}
$$

Before giving the proof of this proposition, we need to specify $P_n(\zeta, z)$.

Lemma 7. *For all $(\zeta, z) \in \mathbb{C}^2$, one can choose*

$$
\begin{aligned}
(5.2) \quad P_n^1(\zeta,z) \quad &= \quad \sum_{u_1=0}^{m_1-1} \zeta_1^{m_1-1-u_1} z_1^{u_1} \prod_{j=2}^{n-1}(z_1-\eta_j z_2)^{m_j} z_2^{m_n} \\
&+ \sum_{p=2}^{n-1}\sum_{u_p=0}^{m_p-1} \zeta_1^{m_1} \prod_{j=2}^{p-1}(\zeta_1-\eta_j\zeta_2)^{m_j} (\zeta_1-\eta_p\zeta_2)^{m_p-1-u_p} (z_1-\eta_p z_2)^{u_p} \prod_{j=p+1}^{n-1} (z_1-\eta_j z_2)^{m_j} z_2^{m_n} \, ,
\end{aligned}
$$

$$
\begin{aligned}
(5.3) \quad P_n^2(\zeta,z) \quad &= \quad \sum_{u_n=0}^{m_n-1} \zeta_1^{m_1} \prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j} \zeta_2^{m_n-1-u_n} z_2^{u_n} \\
&- \sum_{p=2}^{n-1}\sum_{u_p=0}^{m_p-1} \eta_p\, \zeta_1^{m_1} \prod_{j=2}^{p-1}(\zeta_1-\eta_j\zeta_2)^{m_j} (\zeta_1-\eta_p\zeta_2)^{m_p-1-u_p} (z_1-\eta_p z_2)^{u_p} \prod_{j=p+1}^{n-1} (z_1-\eta_j z_2)^{m_j} z_2^{m_n} \, .
\end{aligned}
$$

Proof. First, we prove the following fact by induction on m: $h_1, \ldots, h_m$ being given, consider the associate $P_{h_1}, \ldots, P_{h_m}$. Then one can choose

$$
(5.4) \qquad P_{\prod_{j=1}^m h_j}(\zeta,z) \quad = \quad \sum_{p=1}^{m} \prod_{j=1}^{p-1} h_j(\zeta)\; P_{h_p}(\zeta,z) \prod_{j=p+1}^{m} h_j(z) \, ,
$$

i.e.

$$
\prod_{j=1}^{m} h_j(\zeta) - \prod_{j=1}^{m} h_j(z) \quad = \quad \sum_{p=1}^{m} \prod_{j=1}^{p-1} h_j(\zeta) \prod_{j=p+1}^{m} h_j(z) \; < P_{h_p}(\zeta,z), \zeta - z > \; .
$$

This is obvious for $m=1$. Now consider $h_1,\dots,h_m$, we have

$$\begin{aligned}
\prod_{j=1}^{m} h_j(\zeta) - \prod_{j=1}^{m} h_j(z) &= (h_m(\zeta)-h_m(z)) \prod_{j=1}^{m-1} h_j(\zeta) + \left(\prod_{j=1}^{m-1} h_j(\zeta) - \prod_{j=1}^{m-1} h_j(z)\right) h_m(z) \\
&= \prod_{j=1}^{m-1} h_j(\zeta) < P_{h_m}(\zeta,z), \zeta - z > \\
&\quad + h_m(z) \sum_{p=1}^{m-1} \prod_{j=1}^{p-1} h_j(\zeta) \prod_{j=p+1}^{m-1} h_j(z) < P_{h_p}(\zeta,z), \zeta - z >
\end{aligned}$$

and this proves (5.4).

On the other hand, we see that, for all $\eta \in \mathbb{C}$,

$$P_{(t \mapsto t_1 - \eta t_2)}(\zeta,z) = \begin{pmatrix} 1 \\ -\eta \end{pmatrix},$$

as well as

$$P_{(t \mapsto t_2)}(\zeta,z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

One can deduce from (5.4) the following choice about $g_n(z) = \prod_{u_1=1}^{m_1} z_1 \prod_{p=2}^{n-1} \prod_{u_p=1}^{m_p} (z_1 - \eta_p z_2) \prod_{u_n=1}^{m_n} z_2$:

$$\begin{aligned}
P_n(\zeta,z) \quad = \quad & \sum_{u_1=1}^{m_1} \zeta_1^{u_1-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} z_1^{m_1-u_1} \prod_{j=2}^{n-1} (z_1 - \eta_j z_2)^{m_j} \, z_2^{m_n} \\
& + \sum_{p=2}^{n-1} \sum_{u_p=1}^{m_p} \zeta_1^{m_1} \prod_{j=2}^{p-1} (\zeta_1 - \eta_j \zeta_2)^{m_j} (\zeta_1 - \eta_p \zeta_2)^{u_p-1} \times \\
& \qquad \times \begin{pmatrix} 1 \\ -\eta_p \end{pmatrix} (z_1 - \eta_p z_2)^{m_p-u_p} \prod_{j=p+1}^{n-1} (z_1 - \eta_j z_2)^{m_j} \, z_2^{m_n} \\
& + \sum_{u_n=1}^{m_n} \zeta_1^{m_1} \prod_{j=2}^{n-1} (\zeta_1 - \eta_j \zeta_2)^{m_j} \, \zeta_2^{u_n-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} z_2^{m_n-u_n},
\end{aligned}$$

and the lemma is proved.

$\surd$

Now we can give the proof of proposition 4.

Proof. We have by lemma 7

$$\begin{aligned}\frac{\det(\overline{\zeta}, P_n(\zeta,z))}{g_n(\zeta)} &= \overline{\zeta}_1 \sum_{u_n=0}^{m_n-1} \frac{z_2^{u_n}}{\zeta_2^{u_n+1}} \\ &- \sum_{p=2}^{n-1}\sum_{u_p=0}^{m_p-1} (\overline{\zeta}_2+\eta_p\overline{\zeta}_1)\frac{(z_1-\eta_p z_2)^{u_p}\prod_{j=p+1}^{n-1}(z_1-\eta_j z_2)^{m_j} z_2^{m_n}}{(\zeta_1-\eta_p\zeta_2)^{u_p+1}\prod_{j=p+1}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}\zeta_2^{m_n}} \\ &- \overline{\zeta}_2 \sum_{u_1=0}^{m_1-1} \frac{z_1^{u_1}\prod_{j=2}^{n-1}(z_1-\eta_j z_2)^{m_j} z_2^{m_n}}{\zeta_1^{u_1+1}\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}\zeta_2^{m_n}}\,.\end{aligned}$$

Then we want to calculate, for all $k_1,\ k_2 \geq 0$:

$$\lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2}\int_{\partial\widetilde{\Sigma}_\varepsilon} \frac{\zeta_1^{k_1}\zeta_2^{k_2}\det\left(\overline{\zeta}, P_n(\zeta,z)\right)}{g_n(\zeta)\left(1-<\overline{\zeta},z>\right)}\,\omega(\zeta) \quad = \tag{5.5}$$

$$\begin{aligned} = \quad & \lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2}\left[\int_{r=1-\varepsilon} - \sum_{l=2}^{\widetilde{n}-1}\left(\int_{r=\alpha_{q_l}+\varepsilon} - \int_{r=\alpha_{q_l}-\varepsilon}\right) - \int_{r=\varepsilon}\right] \\ & \frac{\omega(\zeta)}{(\zeta_2-(1-r^2)z_2)(\zeta_1-\frac{r^2 z_1\zeta_2}{\zeta_2-(1-r^2)^2 z_2})} \times \Big\{ \sum_{u_n=0}^{m_n-1} z_2^{u_n} r^2 \zeta_1^{k_1}\zeta_2^{k_2-u_n} \\ - & \sum_{p=2}^{n-1}\sum_{u_p=0}^{m_p-1}(z_1-\eta_p z_2)^{u_p}\prod_{j=p+1}^{n-1}(z_1-\eta_j z_2)^{m_j} z_2^{m_n}\frac{\zeta_1^{k_1}\zeta_2^{k_2-m_n}((1-r^2)\zeta_1+\eta_p r^2\zeta_2)}{(\zeta_1-\eta_p\zeta_2)^{u_p+1}\prod_{j=p+1}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}} \\ - & \sum_{u_1=0}^{m_1-1} z_1^{u_1}\prod_{j=2}^{n-1}(z_1-\eta_j z_2)^{m_j} z_2^{m_n}\frac{(1-r^2)\,\zeta_1^{k_1-u_1}\zeta_2^{k_2-m_n}}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}}\ \Big\}\,.\end{aligned}$$

The proof will be a consequence of lemma 8, lemma 9 and lemma 10.

$\surd$

Lemma 8. *For all $z\in\mathbb{B}_2$, for all $u_n = 0,\dots,m_n-1$ and all $k_1,\ k_2\geq 0$, we have*

$$\lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2}\int_{\partial\widetilde{\Sigma}_\varepsilon}\frac{r^2\zeta_1^{k_1}\zeta_2^{k_2-u_n}\,\omega(\zeta)}{(\zeta_2-(1-r^2)z_2)(\zeta_1-\frac{r^2 z_1\zeta_2}{\zeta_2-(1-r^2)^2 z_2})} \quad = \quad \mathbf{1}_{k_2=u_n}\ z_1^{k_1}\ z_2^{u_n}\,.$$

It follows that

$$\begin{aligned}\sum_{u_n=0}^{m_n-1} z_2^{u_n}\lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2}\int_{\partial\widetilde{\Sigma}_\varepsilon}\frac{r^2\zeta_1^{k_1}\zeta_2^{k_2-u_n}\,\omega(\zeta)}{(\zeta_2-(1-r^2)z_2)(\zeta_1-\frac{r^2 z_1\zeta_2}{\zeta_2-(1-r^2)^2 z_2})} \quad &= \\ &= \quad \mathbf{1}_{k_2\leq m_n-1}\, z_1^{k_1} z_2^{k_2}\,.\end{aligned}$$

Proof. First, for all $l = 2,\dots,\widetilde{n}-1$ and all $u_n\geq 0$

$$\lim_{\varepsilon\to 0}\int_{r=\alpha_{q_l}\pm\varepsilon}\frac{\overline{\zeta}_1\zeta_1^{k_1}\zeta_2^{k_2-u_n}}{(1-<\overline{\zeta},z>)}\,\omega(\zeta) \quad = \quad \int_{r=\alpha_{q_l}}\frac{\overline{\zeta}_1\zeta_1^{k_1}\zeta_2^{k_2-u_n}}{(1-<\overline{\zeta},z>)}\,\omega(\zeta)$$

then

$$\lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2}\int_{\partial\widetilde{\Sigma}_\varepsilon}\frac{\overline{\zeta}_1\zeta_1^{k_1}\zeta_2^{k_2-u_n}}{(1-<\overline{\zeta},z>)}\,\omega(\zeta) \quad =$$

$$= \quad \lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2}\left[\int_{r=1-\varepsilon}-\int_{r=\varepsilon}\right]\frac{r^2\zeta_1^{k_1}\zeta_2^{k_2-u_n}\,\omega(\zeta)}{(\zeta_2-(1-r^2)z_2)(\zeta_1-\frac{r^2z_1\zeta_2}{\zeta_2-(1-r^2)^2z_2})}$$

$$= \quad \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\left[\int_{|\zeta_2|=\sqrt{1-(1-\varepsilon)^2}}-\int_{|\zeta_2|=\sqrt{1-\varepsilon^2}}\right]\frac{\zeta_2^{k_2-u_n}d\zeta_2}{(\zeta_2-(1-r^2)z_2)}\frac{1}{2\pi i}\int_{|\zeta_1|=r}\frac{r^2\zeta_1^{k_1}d\zeta_1}{(\zeta_1-\frac{r^2z_1\zeta_2}{\zeta_2-(1-r^2)^2z_2})}$$

$$= \quad \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\left[\int_{|\zeta_2|=\sqrt{2\varepsilon-\varepsilon^2}}-\int_{|\zeta_2|=\sqrt{1-\varepsilon^2}}\right]\frac{r^2(r^2z_1)^{k_1}\zeta_2^{k_1+k_2-u_n}d\zeta_2}{(\zeta_2-(1-r^2)z_2)^{k_1+1}}\,.$$

This integral is zero if $k_2 < u_n$. It follows that, for all $u_n = 0,\ldots,m_n-1$ and all $k_2 \geq u_n$,

$$\lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2}\int_{\partial\widetilde{\Sigma}_\varepsilon}\frac{\overline{\zeta}_1\zeta_1^{k_1}\zeta_2^{k_2-u_n}}{(1-<\overline{\zeta},z>)}\,\omega(\zeta) \quad =$$

$$= \quad \lim_{\varepsilon\to 0}\frac{(k_1+k_2-u_n)!}{k_1!\,(k_2-u_n)!}z_1^{k_1}z_2^{k_2-u_n}\Big((1-\varepsilon)^{2(k_1+1)}(2\varepsilon-\varepsilon^2)^{k_2-u_n}-\varepsilon^{2(k_1+1)}(1-\varepsilon^2)^{k_2-u_n}\Big)$$

$$= \quad \begin{cases} z_1^{k_1} \text{ if } k_2 = u_n\,, \\ 0 \;\text{ otherwise}\end{cases}$$

and this proves the first part of the lemma.

The second part follows since

$$\sum_{u_n=0}^{m_n-1}\mathbf{1}_{k_2=u_n}z_1^{k_1}z_2^{u_n} \;=\; z_1^{k_1}z_2^{k_n}\sum_{u_n=0}^{m_n-1}\mathbf{1}_{k_2=u_n} \;=\; \mathbf{1}_{k_2\leq m_n-1}\,z_1^{k_1}z_2^{k_n}\,.$$

$\surd$

Next, we have the following result.

Lemma 9. *For all $z\in U_\eta$, for all $p=2,\ldots,n-1$ and all $u_p=0,\ldots,m_p-1$, we have*

$$\lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2}\int_{\partial\widetilde{\Sigma}_\varepsilon}\frac{(\overline{\zeta}_2+\eta_p\overline{\zeta}_1)\,\zeta_1^{k_1}\zeta_2^{k_2-m_n}\,\omega(\zeta)}{(\zeta_1-\eta_p\zeta_2)^{u_p+1}\prod_{j=p+1}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}(1-<\overline{\zeta},z>)} \quad =$$

$$= \quad -\,\mathbf{1}_{k_1+k_2\geq N_{u_p}} \quad \times\Big\{\;\frac{1}{u_p!}\frac{\partial^{u_p}}{\partial t^{u_p}}|_{t=\eta_p}\left[\frac{1+|\eta_p|^2\eta_p/t}{1+|\eta_p|^2}\;\frac{t^{k_1}\left(\frac{z_2+|\eta_p|^2z_1/t}{1+|\eta_p|^2}\right)^{k_1+k_2-N_{u_p}}}{\prod_{j=p+1}^{n-1}(t-\eta_j)^{m_j}}\right]$$

$$+ \quad \sum_{q=p+1}^{n-1}\frac{1}{(m_q-1)!}\frac{\partial^{m_q-1}}{\partial t^{m_q-1}}|_{t=\eta_q}\left[\frac{1+|\eta_q|^2\eta_p/t}{1+|\eta_q|^2}\;\frac{t^{k_1}\left(\frac{z_2+|\eta_p|^2z_1/t}{1+|\eta_p|^2}\right)^{k_1+k_2-N_{u_p}}}{(t-\eta_p)^{u_p+1}\prod_{j=p+1,j\neq q}^{n-1}(t-\eta_j)^{m_j}}\right]\Big\}$$

$$+ \quad \mathbf{1}_{k_2 \leq m_n - 1, k_1 \geq N_{u_p} - k_2}\, \eta_p\, z_1^{k_1 + k_2 - N_{u_p}} \quad \times \Big\{ \frac{1}{u_p!} \frac{\partial^{u_p}}{\partial t^{u_p}} |_{t=\eta_p} \left[\frac{t^{N_{u_p} - 1 - k_2}}{\prod_{j=p+1}^{n-1} (t - \eta_j)^{m_j}} \right]$$

$$+ \sum_{q=p+1}^{n-1} \frac{1}{(m_q - 1)!} \frac{\partial^{m_q - 1}}{\partial t^{m_q - 1}} |_{t=\eta_q} \left[\frac{t^{N_{u_p} - 1 - k_2}}{(t - \eta_p)^{u_p + 1} \prod_{j=p+1, j \neq q}^{n-1} (t - \eta_j)^{m_j}} \right] \Big\} .$$

Proof. First, consider l_p such that $\alpha_{q_{l_p}} = \alpha_p$. Then

$$\lim_{\varepsilon \to 0} \frac{1}{(2\pi i)^2} \int_{\partial \widetilde{\Sigma}_\varepsilon} \frac{(\overline{\zeta}_2 + \eta_p \overline{\zeta}_1)\, \zeta_1^{k_1} \zeta_2^{k_2 - m_n}\, \omega(\zeta)}{(\zeta_1 - \eta_p \zeta_2)^{u_p + 1} \prod_{j=p+1}^{n-1} (\zeta_1 - \eta_j \zeta_2)^{m_j} (1 - < \overline{\zeta}, z >)} = \tag{5.6}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{(2\pi i)^2} \left[\int_{r = 1 - \varepsilon} - \sum_{l = l_p}^{\widetilde{n} - 1} \left(\int_{r = \alpha_{q_l} + \varepsilon} - \int_{r = \alpha_{q_l} - \varepsilon} \right) - \int_{r = \varepsilon} \right]$$

$$\times \frac{(\overline{\zeta}_2 + \eta_p \overline{\zeta}_1)\, \zeta_1^{k_1} \zeta_2^{k_2 - m_n}\, \omega(\zeta)}{(\zeta_1 - \eta_p \zeta_2)^{u_p + 1} \prod_{j=p+1}^{n-1} (\zeta_1 - \eta_j \zeta_2)^{m_j} (1 - < \overline{\zeta}, z >)} .$$

On the other hand, we know by (3.2) in lemma 3 that, for all $r \in [0, 1]$ such that $r \neq \alpha_s$, $\forall\, s = 1, \dots, n-1$ and all $k_1 \geq 0$, we have

$$\frac{1}{2\pi i} \int_{|\zeta_1| = +\infty} \frac{\zeta_1^{k_1} ((1 - r^2)\zeta_1 + \eta_p r^2 \zeta_2)\, d\zeta_1}{(\zeta_1 - \eta_p \zeta_2)^{u_p + 1} \prod_{j=p+1}^{n-1} (\zeta_1 - \eta_j \zeta_2)^{m_j} (\zeta_1 - \frac{r^2 z_1 \zeta_2}{\zeta_2 - (1 - r^2) z_2})} =$$

$$= \mathbf{1}_{k_1 \geq u_p + m_{p+1} + \cdots + m_{n-1}} \times \Big\{ \frac{(\frac{r^2 z_1 \zeta_2}{\zeta_2 - (1 - r^2) z_2})^{k_1} ((1 - r^2) \frac{r^2 z_1 \zeta_2}{\zeta_2 - (1 - r^2) z_2} + \eta_p r^2 \zeta_2)}{(\frac{r^2 z_1 \zeta_2}{\zeta_2 - (1 - r^2) z_2} - \eta_p \zeta_2)^{u_p + 1} \prod_{j=p+1}^{n-1} (\frac{r^2 z_1 \zeta_2}{\zeta_2 - (1 - r^2) z_2} - \eta_j \zeta_2)^{m_j}}$$

$$- \frac{1}{u_p!} \frac{\partial^{u_p}}{\partial \zeta_1^{u_p}} |_{\zeta_1 = \eta_p \zeta_2} \left[\frac{\zeta_1^{k_1} ((1 - r^2)\zeta_1 + \eta_p r^2 \zeta_2)}{\prod_{j=p+1}^{n-1} (\zeta_1 - \eta_j \zeta_2)^{m_j} (\frac{r^2 z_1 \zeta_2}{\zeta_2 - (1 - r^2) z_2} - \zeta_1)} \right]$$

$$- \sum_{v=p+1}^{n-1} \frac{1}{(m_v - 1)!} \times$$

$$\frac{\partial^{m_v - 1}}{\partial \zeta_1^{m_v - 1}} |_{\zeta_1 = \eta_v \zeta_2} \left[\frac{\zeta_1^{k_1} ((1 - r^2)\zeta_1 + \eta_p r^2 \zeta_2)}{(\zeta_1 - \eta_p \zeta_2)^{u_p + 1} \prod_{j=p+1, j \neq v}^{n-1} (\zeta_1 - \eta_j \zeta_2)^{m_j} (\frac{r^2 z_1 \zeta_2}{\zeta_2 - (1 - r^2) z_2} - \zeta_1)} \right] \Big\}$$

$$= \mathbf{1}_{k_1 \geq u_p + m_{p+1} + \cdots + m_{n-1}}\, \zeta_2^{k_1 - (u_p + m_{p+1} + \cdots + m_{n-1})}\, P_r \left(\frac{r^2 z_1}{\zeta_2 - (1 - r^2) z_2} \right) , \tag{5.7}$$

with the following quotient

$$P_r(X) = Q \left(X^{k_1} \left((1 - r^2) X + \eta_p r^2 \right) ,\, (X - \eta_p)^{u_p + 1} \prod_{j=p+1}^{n-1} (X - \eta_j)^{m_j} \right) .$$

It follows that, for all $l = l_p, \dots, \widetilde{n} - 1$ and for all $r = \alpha_{q_l} + \varepsilon$ (in particular, $r > \alpha_p$),

$$\frac{1}{2\pi i} \int_{|\zeta_1| = r} \frac{\zeta_1^{k_1} ((1 - r^2)\zeta_1 + \eta_p r^2 \zeta_2)\, d\zeta_1}{(\zeta_1 - \eta_p \zeta_2)^{u_p + 1} \prod_{j=p+1}^{n-1} (\zeta_1 - \eta_j \zeta_2)^{m_j} (\zeta_1 - \frac{r^2 z_1 \zeta_2}{\zeta_2 - (1 - r^2) z_2})} =$$

$$
\begin{aligned}
= \quad & \mathbf{1}_{k_1 \geq u_p+m_{p+1}+\cdots+m_{n-1}}\, \zeta_2^{k_1-(u_p+m_{p+1}+\cdots+m_{n-1})}\, P_r\left(\frac{r^2 z_1}{\zeta_2-(1-r^2)z_2}\right) \\
- \quad & \sum_{v\leq n-1, \alpha_v>\alpha_{q_l}} \frac{1}{(m_v-1)!} \times \\
& \frac{\partial^{m_v-1}}{\partial \zeta_1^{m_v-1}}|_{\zeta_1=\eta_v\zeta_2} \frac{\zeta_1^{k_1}((1-r^2)\zeta_1+\eta_p r^2\zeta_2)}{(\zeta_1-\eta_p\zeta_2)^{u_p+1}\prod_{j=p+1, j\neq v}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}(\zeta_1-\frac{r^2 z_1\zeta_2}{\zeta_2-(1-r^2)z_2})}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \zeta_2^{k_1-(u_p+m_{p+1}+\cdots+m_{n-1})} \times \{\, \mathbf{1}_{k_1 \geq u_p+m_{p+1}+\cdots+m_{n-1}}\, P_r\left(\frac{r^2 z_1}{\zeta_2-(1-r^2)z_2}\right) \\
- \quad & \sum_{v\leq n-1, \alpha_v>\alpha_{q_l}} \frac{1}{(m_v-1)!} \times \\
& \frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v} \frac{t^{k_1}((1-r^2)t+\eta_p r^2)}{(t-\eta_p)^{u_p+1}\prod_{j=p+1, j\neq v}^{n-1}(t-\eta_j)^{m_j}(t-\frac{r^2 z_1}{\zeta_2-(1-r^2)z_2})}\ \}\,.
\end{aligned}
$$

Then

$$
\frac{1}{(2\pi i)^2}\int_{r=\alpha_{q_l}+\varepsilon} \frac{(\overline{\zeta}_2+\eta_p\overline{\zeta}_1)\,\zeta_1^{k_1}\zeta_2^{k_2-m_n}\,\omega(\zeta)}{(\zeta_1-\eta_p\zeta_2)^{u_p+1}\prod_{j=p+1}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}(1-<\overline{\zeta},z>)} \quad =
$$

$$
\begin{aligned}
= \quad & \mathbf{1}_{k_1 \geq u_p+m_{p+1}+\cdots+m_{n-1}} \frac{1}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}} \frac{\zeta_2^{k_1+k_2-N_{u_p}}}{\zeta_2-(1-r^2)z_2} P_r\left(\frac{r^2 z_1}{\zeta_2-(1-r^2)z_2}\right) d\zeta_2 \\
- \quad & \sum_{v\leq n-1, \alpha_v>\alpha_{q_l}} \frac{1}{(m_v-1)!}\frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v} \frac{t^{k_1}((1-r^2)+\eta_p r^2/t)}{(t-\eta_p)^{u_p+1}\prod_{j=p+1, j\neq v}^{n-1}(t-\eta_j)^{m_j}} \times \\
& \qquad \times \frac{1}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}} \frac{\zeta_2^{k_1+k_2-N_{u_p}}}{\zeta_2-(1-r^2)z_2-r^2 z_1/t}\, d\zeta_2\ .
\end{aligned}
$$

Similarly, for all $l = l_p, \ldots, \widetilde{n}-1$ and all $r = \alpha_{q_l}-\varepsilon$,

$$
\frac{1}{(2\pi i)^2}\int_{r=\alpha_{q_l}-\varepsilon} \frac{(\overline{\zeta}_2+\eta_p\overline{\zeta}_1)\,\zeta_1^{k_1}\zeta_2^{k_2-m_n}\,\omega(\zeta)}{(\zeta_1-\eta_p\zeta_2)^{u_p+1}\prod_{j=p+1, j\neq v}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}(1-<\overline{\zeta},z>)} \quad =
$$

$$
\begin{aligned}
= \quad & \mathbf{1}_{k_1 \geq u_p+m_{p+1}+\cdots+m_{n-1}} \frac{1}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}} \frac{\zeta_2^{k_1+k_2-N_{u_p}}}{\zeta_2-(1-r^2)z_2} P_r\left(\frac{r^2 z_1}{\zeta_2-(1-r^2)z_2}\right) d\zeta_2 \\
- \quad & \sum_{v\leq n-1, \alpha_v\geq\alpha_l, v\neq p} \frac{1}{(m_v-1)!}\frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v} \frac{t^{k_1}((1-r^2)+\eta_p r^2/t)}{(t-\eta_p)^{u_p+1}\prod_{j=p+1, j\neq v}^{n-1}(t-\eta_j)^{m_j}} \times \\
& \qquad \times \frac{1}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}} \frac{\zeta_2^{k_1+k_2-N_{u_p}}}{\zeta_2-(1-r^2)z_2-r^2 z_1/t}\, d\zeta_2 \\
- \quad & \mathbf{1}_{l=l_p}\, \frac{1}{u_p!}\frac{\partial^{u_p}}{\partial t^{u_p}}|_{t=\eta_p} \frac{t^{k_1}((1-r^2)+\eta_p r^2/t)}{\prod_{j=p+1}^{n-1}(t-\eta_j)^{m_j}} \frac{1}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}} \frac{\zeta_2^{k_1+k_2-N_{u_p}}}{\zeta_2-(1-r^2)z_2-r^2 z_1/t} d\zeta_2\,.
\end{aligned}
$$

Now we know by lemma 6 that, for all $z \in \mathbb{B}_2$, for all r close to $\alpha_s,\ s = 2, \dots, n-1$ and all t close to η_s, one has $|(1-r^2)z_2 + r^2 z_1/t| < \sqrt{1-r^2}$. It follows that

$$\lim_{\varepsilon\to 0}\sum_{l=l_p}^{\tilde{n}-1}\left(\int_{r=\alpha_{q_l}+\varepsilon} - \int_{r=\alpha_{q_l}-\varepsilon}\right) \quad = \quad \mathbf{1}_{k_1+k_2\geq N_{u_p}} \sum_{l=l_p}^{\tilde{n}-1}\sum_{p+1\leq v\leq n-1, \alpha_v=\alpha_{q_l}} \frac{1}{(m_v-1)!}\times$$

$$\times\frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v}\left[\frac{t^{k_1}((1-\alpha_{q_l}^2)+\eta_p\alpha_{q_l}^2/t)}{(t-\eta_p)^{u_p+1}\prod_{j=p+1,j\neq p}^{n-1}(t-\eta_j)^{m_j}}\left((1-\alpha_{q_l}^2)z_2+\alpha_{q_l}^2 z_1/t\right)^{k_1+k_2-N_{u_p}}\right]$$

$$+\,\mathbf{1}_{k_1+k_2\geq N_{u_p}}\frac{1}{u_p!}\frac{\partial^{u_p}}{\partial t^{u_p}}|_{t=\eta_p}\left[\frac{t^{k_1}((1-\alpha_{q_{l_p}}^2)+\eta_p\alpha_{q_{l_p}}^2/t)}{\prod_{j=p+1}^{n-1}(t-\eta_j)^{m_j}}\left((1-\alpha_{q_{l_p}}^2)z_2+\alpha_{q_{l_p}}^2 z_1/t\right)^{k_1+k_2-N_{u_p}}\right]$$

$$(5.8)\quad = \mathbf{1}_{k_1+k_2\geq N_{u_p}}\ \{\sum_{v=p+1}^{n-1}\frac{1}{(m_v-1)!}\times$$

$$\times\ \frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v}\left[\frac{t^{k_1}((1-\alpha_v^2)+\eta_p\alpha_v^2/t)}{(t-\eta_p)^{u_p+1}\prod_{j=p+1,j\neq p}^{n-1}(t-\eta_j)^{m_j}}\left((1-\alpha_v^2)z_2+\alpha_v^2 z_1/t\right)^{k_1+k_2-N_{u_p}}\right]$$

$$+\quad \frac{1}{u_p!}\frac{\partial^{u_p}}{\partial t^{u_p}}|_{t=\eta_p}\left[\frac{t^{k_1}((1-\alpha_p^2)+\eta_p\alpha_p^2/t)}{\prod_{j=p+1}^{n-1}(t-\eta_j)^{m_j}}\left((1-\alpha_p^2)z_2+\alpha_p^2 z_1/t\right)^{k_1+k_2-N_{u_p}}\right]\}\,.$$

Notice that we used the continuity on r of $P_r(X)$. One has else $X^{k_1}\left((1-r^2)X+\eta_p r^2\right) = (1-r^2)X^{k_1+1}+\eta_p r^2 X^{k_1}$ and one can separately deal with each integral as above.

On the other hand, if $k_1 < u_p + \cdots + m_{n-1}$ or $k_1 + k_2 < N_{u_p}$, we have

$$\frac{1}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}}\frac{\zeta_2^{k_1+k_2-N_{u_p}}}{\zeta_2-(1-r^2)z_2}P_r\left(\frac{r^2 z_1}{\zeta_2-(1-r^2)z_2}\right)d\zeta_2 \quad = \quad 0\,.$$

It follows by (5.7) that, for all $k_1 \geq u_p + \cdots + m_{n-1}$, for all $k_1 + k_2 \geq N_{u_p}$, for all $\varepsilon > 0$ small enough and all $z \in U_\eta$,

$$\frac{1}{(2\pi i)^2}\int_{r=1-\varepsilon}\frac{(\overline{\zeta}_2+\eta_p\overline{\zeta}_1)\,\zeta_1^{k_1}\zeta_2^{k_2-m_n}\,\omega(\zeta)}{(\zeta_1-\eta_p\zeta_2)^{u_p+1}\prod_{j=p+1}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}(1-<\overline{\zeta},z>)} \quad =$$

$$=\quad \frac{1}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}}\frac{\zeta_2^{k_2-m_n}}{\zeta_2-(1-r^2)z_2}\,d\zeta_2\ \times$$

$$\times\ \frac{1}{2\pi i}\int_{|\zeta_1|=+\infty}\frac{\zeta_1^{k_1}\left((1-r^2)\zeta_1+\eta_p r^2\zeta_2\right)\,d\zeta_1}{(\zeta_1-\eta_p\zeta_2)^{u_p+1}\prod_{j=p+1}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}\left(\zeta_1-\frac{r^2 z_1\zeta_2}{\zeta_2-(1-r^2)z_2}\right)}$$

$$=\quad \frac{1}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}}\frac{\zeta_2^{k_1+k_2-N_{u_p}}}{\zeta_2-(1-r^2)z_2}P_r\left(\frac{r^2 z_1}{\zeta_2-(1-r^2)z_2}\right)d\zeta_2$$

$$= \lim_{\varepsilon'\to 0}\frac{1}{2\pi i}\int_{|\zeta_2-(1-r^2)z_2|=\varepsilon'}\frac{\zeta_2^{k_1+k_2-N_{u_p}}}{\zeta_2-(1-r^2)z_2}\times$$

$$\times\frac{\left(\frac{r^2z_1}{\zeta_2-(1-r^2)z_2}\right)^{k_1}\left(\frac{(1-r^2)r^2z_1}{\zeta_2-(1-r^2)z_2}+\eta_p r^2\right)d\zeta_2}{\left(\frac{r^2z_1}{\zeta_2-(1-r^2)z_2}-\eta_p\right)^{u_p+1}\prod_{j=p+1}^{n-1}\left(\frac{r^2z_1}{\zeta_2-(1-r^2)z_2}-\eta_j\right)^{m_j}}$$

$$+\quad \frac{1}{u_p!}\frac{\partial^{u_p}}{\partial t^{u_p}}|_{t=\eta_p}\frac{t^{k_1}((1-r^2)t+\eta_p r^2)}{\prod_{j=p+1}^{n-1}(t-\eta_j)^{m_j}}\lim_{\varepsilon'\to 0}\frac{1}{2\pi i}\int_{|\zeta_2-(1-r^2)z_2|=\varepsilon'}\frac{\zeta_2^{k_1+k_2-N_{u_p}}}{t(\zeta_2-(1-r^2)z_2)-r^2z_1}d\zeta_2$$

$$+\quad \sum_{v=p+1}^{n-1}\frac{1}{(m_v-1)!}\frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v}\frac{t^{k_1}((1-r^2)t+\eta_p r^2)}{(t-\eta_p)^{u_p+1}\prod_{j=p+1,j\neq p}^{n-1}(t-\eta_j)^{m_j}}\times$$

$$\times\lim_{\varepsilon'\to 0}\frac{1}{2\pi i}\int_{|\zeta_2-(1-r^2)z_2|=\varepsilon'}\frac{\zeta_2^{k_1+k_2-N_{u_p}}}{t(\zeta_2-(1-r^2)z_2)-r^2z_1}d\zeta_2$$

$$= r^2(r^2z_1)^{k_1}\lim_{\varepsilon'\to 0}\frac{1}{2\pi i}\int_{|\zeta_2-(1-r^2)z_2|=\varepsilon'}\frac{d\zeta_2}{(\zeta_2-(1-r^2)z_2)^{k_1-(u_p+\cdots+m_{n-1})+1}}\times$$

$$\times\frac{\zeta_2^{k_1+k_2-N_{u_p}}\left((1-r^2)z_1+\eta_p(\zeta_2-(1-r^2)z_2)\right)}{(r^2z_1-\eta_p(\zeta_2-(1-r^2)z_2))^{u_p+1}\prod_{j=p+1}^{n-1}(r^2z_1-\eta_j(\zeta_2-(1-r^2)z_2))^{m_j}}$$

$$+\quad 0$$

$$= r^2(r^2z_1)^{k_1}\sum_{v_p+\cdots+v_n=k_1-(u_p+\cdots+m_{n-1})}\frac{(u_p+v_p)!}{u_p!\,v_p!}\times$$

$$\times\frac{\eta_p^{v_p}}{(r^2z_1)^{u_p+v_p+1}}\prod_{j=p+1}^{n-1}\frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\frac{\eta_j^{v_j}}{(r^2z_1)^{v_j+m_j}}$$

$$\times\frac{1}{v_n!}\frac{\partial^{v_n}}{\partial\zeta_2^{v_n}}|_{\zeta_2=(1-r^2)z_2}\left(\zeta_2^{k_1+k_2-N_{u_p}}\left(\eta_p\zeta_2+(1-r^2)(z_1-\eta_p z_2)\right)\right)$$

$$\xrightarrow[r=1-\varepsilon\to 1]{}\sum_{v_p+\cdots+v_n=k_1-(u_p+\cdots+m_{n-1}),v_n\geq 1}\frac{(u_p+v_p)!}{u_p!\,v_p!}\eta_p^{v_p}\prod_{j=p+1}^{n-1}\frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\eta_j^{v_j}$$

$$\times\quad\frac{(k_1+k_2-N_{u_p}+1)!}{v_n!\,(k_1+k_2-N_{u_p}+1-v_n)!}\eta_p\,z_1^{v_n-1}(0)^{k_1+k_2-N_{u_p}+1-v_n}$$

$$= \mathbf{1}_{k_2\leq m_n-1}\,\eta_p\,z_1^{k_1+k_2-N_{u_p}}$$

$$\times\sum_{v_p+\cdots+v_{n-1}=m_n-1-k_2}\frac{(u_p+v_p)!}{u_p!\,v_p!}\eta_p^{v_p}\prod_{j=p+1}^{n-1}\frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\eta_j^{v_j}$$

(since if $k_2\geq m_n$ then $k_1+k_2-(u_p+\cdots+m_n)+1-v_n\geq k_1-(u_p+\cdots+m_{n-1})-v_n+1\geq 1$). By lemma 3, this gives

$$\mathbf{1}_{k_2\leq m_n-1}\,\eta_p\,z_1^{k_1+k_2-N_{u_p}}Q\left(X^{N_{u_p}-k_2},(X-\eta_p)^{u_p+1}\prod_{j=p+1}^{n-1}(X-\eta_j)^{m_j}\right)|_{X=0}$$

$$= \mathbf{1}_{k_2 \le m_n - 1}\, \eta_p\, z_1^{k_1+k_2-N_{u_p}} \times$$
$$\times \left[\frac{X^{N_{u_p}-k_2} - R\left(X^{N_{u_p}-k_2}, (X-\eta_p)^{u_p+1} \prod_{j=p+1}^{n-1} (X-\eta_j)^{m_j} \right)}{(X-\eta_p)^{u_p+1} \prod_{j=p+1}^{n-1} (X-\eta_j)^{m_j}} \right] |_{X=0}$$

$$(5.9) \quad = \mathbf{1}_{k_2 \le m_n - 1}\, \eta_p\, z_1^{k_1+k_2-N_{u_p}} \times \Big\{ \frac{1}{u_p!} \frac{\partial^{u_p}}{\partial t^{u_p}} |_{t=\eta_p} \left[\frac{t^{N_{u_p}-k_2-1}}{\prod_{j=p+1}^{n-1} (t-\eta_j)^{m_j}} \right]$$
$$+ \sum_{q=p+1}^{n-1} \frac{1}{(m_q-1)!} \frac{\partial^{m_q-1}}{\partial t^{m_q-1}} |_{t=\eta_q} \left[\frac{t^{N_{u_p}-k_2-1}}{(t-\eta_p)^{u_p+1} \prod_{j=p+1, j\neq q}^{n-1} (t-\eta_j)^{m_j}} \right] \Big\}$$

Similarly, we have

$$\frac{1}{(2\pi i)^2} \int_{r=\varepsilon} \frac{(\overline{\zeta}_2 + \eta_p \overline{\zeta}_1)\, \zeta_1^{k_1} \zeta_2^{k_2-m_n}\, \omega(\zeta)}{(\zeta_1 - \eta_p \zeta_2)^{u_p+1} \prod_{j=p+1}^{n-1} (\zeta_1 - \eta_j \zeta_2)^{m_j} (1 - < \overline{\zeta}, z >)} =$$

$$= \mathbf{1}_{k_1 \ge u_p + \cdots + m_{n-1}} \frac{1}{2\pi i} \int_{|\zeta_2| = \sqrt{1-r^2}} \frac{\zeta_2^{k_1+k_2-N_{u_p}}}{\zeta_2 - (1-r^2) z_2} P_r \left(\frac{r^2 z_1}{\zeta_2 - (1-r^2) z_2} \right) d\zeta_2$$
$$- \frac{1}{u_p!} \frac{\partial^{u_p}}{\partial t^{u_p}} |_{t=\eta_p} \frac{t^{k_1} ((1-r^2) + \eta_p r^2/t)}{\prod_{j=p+1}^{n-1} (t-\eta_j)^{m_j}} \frac{1}{2\pi i} \int_{|\zeta_2| = \sqrt{1-r^2}} \frac{\zeta_2^{k_1+k_2-N_{u_p}}}{\zeta_2 - (1-r^2) z_2 - r^2 z_1/t} d\zeta_2$$
$$- \sum_{v=p+1}^{n-1} \frac{1}{(m_v-1)!} \frac{\partial^{m_v-1}}{\partial t^{m_v-1}} |_{t=\eta_v} \frac{t^{k_1} ((1-r^2) + \eta_p r^2/t)}{(t-\eta_p)^{u_p+1} \prod_{j=p+1, j\neq v}^{n-1} (t-\eta_j)^{m_j}} \times$$
$$\times \frac{1}{2\pi i} \int_{|\zeta_2| = \sqrt{1-r^2}} \frac{\zeta_2^{k_1+k_2-N_{u_p}}}{\zeta_2 - (1-r^2) z_2 - r^2 z_1/t} d\zeta_2$$

$$= \mathbf{1}_{k_1 \ge u_p + \cdots + m_{n-1},\, k_1+k_2 \ge N_{u_p}} \times$$
$$\times \sum_{v_p + \cdots + v_n = k_1 - (u_p + \cdots + m_{n-1})} \frac{(u_p+v_p)!}{u_p!\, v_p!} \eta_p^{v_p} \prod_{j=p+1}^{n-1} \frac{(v_j + m_j - 1)!}{v_j!\, (m_j-1)!} \eta_j^{v_j}$$
$$\times \; r^2 (r^2 z_1)^{v_n - 1} \frac{1}{v_n!} \frac{\partial^{v_n}}{\partial \zeta_2^{v_n}} |_{\zeta_2 = (1-r^2) z_2} \left(\zeta_2^{k_1+k_2-N_{u_p}} \left(\eta_p \zeta_2 + (1-r^2)(z_1 - \eta_p z_2) \right) \right)$$

$$- \mathbf{1}_{k_1+k_2 \ge N_{u_p}} \frac{1}{u_p!} \frac{\partial^{u_p}}{\partial t^{u_p}} |_{t=\eta_p} \frac{t^{k_1} ((1-r^2) + \eta_p r^2/t)}{\prod_{j=p+1}^{n-1} (t-\eta_j)^{m_j}} \left((1-r^2) z_2 + r^2 z_1/t \right)^{k_1+k_2-N_{u_p}}$$
$$- \mathbf{1}_{k_1+k_2 \ge N_{u_p}} \sum_{v=p+1}^{n-1} \frac{1}{(m_v-1)!} \times$$
$$\times \frac{\partial^{m_v-1}}{\partial t^{m_v-1}} |_{t=\eta_v} \frac{t^{k_1} ((1-r^2) + \eta_p r^2/t)}{(t-\eta_p)^{u_p+1} \prod_{j=p+1, j\neq v}^{n-1} (t-\eta_j)^{m_j}} \left((1-r^2) z_2 + r^2 z_1/t \right)^{k_1+k_2-N_{u_p}}$$

$$\begin{aligned}
\xrightarrow[r=\varepsilon\to 0]{} \quad & \mathbf{1}_{k_1+k_2\geq N_{u_p}} \sum_{v_p+\cdots+v_n=k_1-(u_p+\cdots+m_{n-1}),v_n=0} \frac{(u_p+v_p)!}{u_p!\,v_p!}\eta_p^{v_p}\prod_{j=p+1}^{n-1}\frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\eta_j^{v_j} \\
& \times 1 \times z_2^{k_1+k_2-N_{u_p}} \\
& - \mathbf{1}_{k_1+k_2\geq N_{u_p}} z_2^{k_1+k_2-N_{u_p}} \times \Big\{ \frac{1}{u_p!}\frac{\partial^{u_p}}{\partial t^{u_p}}|_{t=\eta_p} \frac{t^{k_1}}{\prod_{j=p+1}^{n-1}(t-\eta_j)^{m_j}} \\
& + \sum_{v=p+1}^{n-1} \frac{1}{(m_v-1)!}\frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v} \frac{t^{k_1}}{(t-\eta_p)^{u_p+1}\prod_{j=p+1,j\neq v}^{n-1}(t-\eta_j)^{m_j}} \Big\}
\end{aligned}$$

$$(5.10) \quad = \quad 0\,,$$

the last equality coming from lemma 3 since

$$\begin{aligned}
& \sum_{v_p+\cdots+v_{n-1}=k_1+1-(u_p+1+m_{p+1}+\cdots+m_{n-1})} \frac{(u_p+v_p)!}{u_p!\,v_p!}\eta_p^{v_p}\prod_{j=p+1}^{n-1}\frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\eta_j^{v_j} \\
& \qquad + \frac{1}{u_p!}\frac{\partial^{u_p}}{\partial t^{u_p}}|_{t=\eta_p}\frac{t^{k_1+1}}{\prod_{j=p+1}^{n-1}(t-\eta_j)^{m_j}(-t)} \\
& + \sum_{v=p+1}^{n-1}\frac{1}{(m_v-1)!}\frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v}\frac{t^{k_1+1}}{(t-\eta_p)^{u_p+1}\prod_{j=p+1,j\neq v}^{n-1}(t-\eta_j)^{m_j}(-t)} \quad =
\end{aligned}$$

$$= \left[Q\left(X^{k_1+1}, (X-\eta_p)^{u_p+1}\prod_{j=p+1}^{n-1}(X-\eta_j)^{m_j}\right) + \frac{R\left(X^{k_1+1},(X-\eta_p)^{u_p+1}\prod_{j=p+1}^{n-1}(X-\eta_j)^{m_j}\right)}{(X-\eta_p)^{u_p+1}\prod_{j=p+1}^{n-1}(X-\eta_j)^{m_j}}\right]|_{X=0}$$

$$\begin{aligned}
= \quad & \left[\frac{X^{k_1+1}}{(X-\eta_p)^{u_p+1}\prod_{j=p+1}^{n-1}(X-\eta_j)^{m_j}}\right]|_{X=0} \\
= \quad & 0\,.
\end{aligned}$$

Finally, by (5.6), (5.8), (5.9), and (5.10), we get for all $k_1+k_2\geq N_{u_p}$ (else we get zero) and all $z\in U_\eta$

$$\lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2}\int_{\partial\widetilde{\Sigma}_\varepsilon}\frac{(\overline{\zeta}_2+\eta_p\overline{\zeta}_1)\,\zeta_1^{k_1}\zeta_2^{k_2-m_n}\,\omega(\zeta)}{(\zeta_1-\eta_p\zeta_2)^{u_p+1}\prod_{j=p+1}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}(1-<\overline{\zeta},z>)} \quad =$$

$$\begin{aligned}
= \quad & \mathbf{1}_{k_2\leq m_n-1}\,\eta_p\,z_1^{k_1+k_2-N_{u_p}} \times \Big\{ \frac{1}{u_p!}\frac{\partial^{u_p}}{\partial t^{u_p}}|_{t=\eta_p}\left[\frac{t^{N_{u_p}-k_2-1}}{\prod_{j=p+1}^{n-1}(t-\eta_j)^{m_j}}\right] \\
& + \sum_{q=p+1}^{n-1}\frac{1}{(m_q-1)!}\frac{\partial^{m_q-1}}{\partial t^{m_q-1}}|_{t=\eta_q}\left[\frac{t^{N_{u_p}-k_2-1}}{(t-\eta_p)^{u_p+1}\prod_{j=p+1,j\neq q}^{n-1}(t-\eta_j)^{m_j}}\right]\Big\}
\end{aligned}$$

$$
\begin{aligned}
- & \quad \frac{1}{u_p!}\frac{\partial^{u_p}}{\partial t^{u_p}}|_{t=\eta_p}\left[\frac{t^{k_1}\,\frac{1+|\eta_p|^2\eta_p/t}{1+|\eta_p|^2}}{\prod_{j=p+1}^{n-1}(t-\eta_j)^{m_j}}\left(\frac{z_2+|\eta_p|^2 z_1/t}{1+|\eta_p|^2}\right)^{k_1+k_2-N_{u_p}}\right] \\
- & \quad \sum_{v=p+1}^{n-1}\frac{1}{(m_v-1)!}\frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v}\left[\frac{t^{k_1}\,\frac{1+|\eta_v|^2\eta_p/t}{1+|\eta_v|^2}\left(\frac{z_2+|\eta_v|^2 z_1/t}{1+|\eta_v|^2}\right)^{k_1+k_2-N_{u_p}}}{(t-\eta_p)^{u_p+1}\prod_{j=p+1,j\neq p}^{n-1}(t-\eta_j)^{m_j}}\right] \\
- & \quad 0
\end{aligned}
$$

and the lemma is proved.

$\surd$

Lastly, we prove the following lemma.

Lemma 10. *For all $z\in U_\eta$ and all $u_1=0,\ldots,m_1-1$, we have*

$$
\lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2}\int_{\partial\widetilde{\Sigma}_\varepsilon}\frac{\overline{\zeta}_2\,\zeta_2^{k_2-m_n}\zeta_1^{k_1-u_1-1}\;\omega(\zeta)}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}\,(1-<\overline{\zeta},z>)} \quad =
$$

$$
\begin{aligned}
= & \quad \mathbf{1}_{k_1+k_2\geq N_{u_1}}\sum_{p=2}^{n-1}\frac{1}{(m_p-1)!}\frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p}\,\frac{1}{t^{u_1+1}\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}}\times \\
& \times\{\,\mathbf{1}_{k_1\leq u_1}\;t^{k_1}z_2^{k_1+k_2-N_{u_1}}-\frac{t^{k_1}}{1+|\eta_p|^2}\left(\frac{z_2+|\eta_p|^2 z_1/t}{1+|\eta_p|^2}\right)^{k_1+k_2-N_{u_1}}\quad\}\,.
\end{aligned}
$$

Proof. The proof is analogous to the one of the previous lemma. We want to calculate

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2} & \left[\int_{r=1-\varepsilon}-\sum_{l=2}^{\widetilde{n}-1}\left(\int_{r=\alpha_{q_l}+\varepsilon}-\int_{r=\alpha_{q_l}-\varepsilon}\right)-\int_{r=\varepsilon}\right] \\
& \times\frac{(1-r^2)\,\zeta_2^{k_2-m_n}}{\zeta_2-(1-r^2)z_2}\;\frac{\zeta_1^{k_1-u_1}}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}\left(\zeta_1-\frac{r^2 z_1\zeta_2}{\zeta_2-(1-r^2)z_2}\right)}\,\omega(\zeta)\,.
\end{aligned}
$$

First, we have similarly

$$
\frac{1}{2\pi i}\int_{|\zeta_1|=+\infty}\frac{\zeta_1^{k_1-u_1}}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}\left(\zeta_1-\frac{r^2 z_1\zeta_2}{\zeta_2-(1-r^2)z_2}\right)}\,d\zeta_1 \quad = \tag{5.11}
$$

$$
= \quad \mathbf{1}_{k_1\geq u_1+m_2+\cdots+m_{n-1}}\,\zeta_2^{k_1-(u_1+m_2+\cdots+m_{n-1})}P\left(\frac{r^2 z_1}{\zeta_2-(1-r^2)z_2}\right)\,,
$$

with the following quotient

$$P(X) \quad = \quad Q\left(X^{k_1-u_1}, \prod_{j=2}^{n-1}(X-\eta_j)^{m_j}\right)$$

(notice that if $k_1 \geq u_1 + m_2 + \cdots + m_{n-1}$ then in particular $k_1 \geq u_1$).

It follows that, for all $l = 1, \ldots, \widetilde{n}-1$ and for all small enough $\varepsilon > 0$, we have

$$\frac{1}{2\pi i}\int_{r=\alpha_{q_l}+\varepsilon} \frac{\zeta_1^{k_1-u_1}}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}\left(\zeta_1-\frac{r^2 z_1\zeta_2}{\zeta_2-(1-r^2)z_2}\right)}\,d\zeta_1 \quad =$$

$$\begin{aligned} &= \quad \mathbf{1}_{k_1\geq u_1+m_2+\cdots+m_{n-1}}\,\zeta_2^{k_1-(u_1+m_2+\cdots+m_{n-1})}P\left(\frac{r^2 z_1}{\zeta_2-(1-r^2)z_2}\right) \\ &- \quad \zeta_2^{k_1-(u_1+m_2+\cdots+m_{n-1})}\sum_{v\leq n-1,\alpha_v>\alpha_{q_l}}\frac{1}{(m_v-1)!}\ \times \\ &\qquad \times\ \frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v}\left[\frac{t^{k_1-u_1}}{\prod_{j=2,j\neq v}^{n-1}(t-\eta_j)^{m_j}\left(t-\frac{r^2 z_1}{\zeta_2-(1-r^2)z_2}\right)}\right] \end{aligned}$$

then by lemma 6, for $r = \alpha_{q_l}+\varepsilon$ and all $z \in U_\eta$,

$$\frac{1}{(2\pi i)^2}\int_{r=\alpha_{q_l}+\varepsilon}\frac{(1-r^2)\,\zeta_2^{k_2-m_n}}{\zeta_2-(1-r^2)z_2}\ \frac{\zeta_1^{k_1-u_1}}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}\left(\zeta_1-\frac{r^2 z_1\zeta_2}{\zeta_2-(1-r^2)z_2}\right)}\,\omega(\zeta) \quad =$$

$$\begin{aligned} &= \ \mathbf{1}_{k_1\geq u_1+m_2+\cdots+m_{n-1}}\frac{1-r^2}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}}\frac{\zeta_2^{k_1+k_2-N_{u_1}}}{\zeta_2-(1-r^2)z_2}P\left(\frac{r^2 z_1}{\zeta_2-(1-r^2)z_2}\right)d\zeta_2 \\ &\quad - \quad \sum_{v\leq n-1,\alpha_v>\alpha_{q_l}}\frac{1-r^2}{(m_v-1)!}\ \times \\ &\qquad \frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v}\frac{t^{k_1-u_1-1}}{\prod_{j=2,j\neq v}^{n-1}(t-\eta_j)^{m_j}}\frac{1}{2\pi i}\int_{|\zeta_2|=\sqrt{1-r^2}}\frac{\zeta_2^{k_1+k_2-N_{u_1}}\,d\zeta_2}{\zeta_2-(1-r^2)z_2-r^2 z_1/t} \end{aligned}$$

$$\begin{aligned} &= \quad \mathbf{1}_{k_1\geq u_1+m_2+\cdots+m_{n-1},\,k_1+k_2\geq N_{u_1}}\ (1-r^2)\ \times \\ &\{\lim_{\varepsilon'\to 0}\frac{1}{2\pi i}\int_{|\zeta_2-(1-r^2)z_2|=\varepsilon'}\frac{\zeta_2^{k_1+k_2-N_{u_1}}(r^2 z_1)^{k_1-u_1}\ d\zeta_2}{(\zeta_2-(1-r^2)z_2)^{k_1-(u_1+\cdots+m_{n-1})+1}\prod_{j=2}^{n-1}\left(r^2 z_1-\eta_j(\zeta_2-(1-r^2)z_2)\right)^{m_j}} \\ &\qquad + 0\ \} \\ &\quad - \quad \mathbf{1}_{k_1+k_2\geq N_{u_1}}\sum_{v\leq n-1,\alpha_v>\alpha_{q_l}}\frac{1-r^2}{(m_v-1)!}\ \times \\ &\qquad \times\ \frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v}\left[\frac{t^{k_1-u_1-1}}{\prod_{j=2,j\neq v}^{n-1}(t-\eta_j)^{m_j}}\left((1-r^2)z_2+r^2 z_1/t\right)^{k_1+k_2-N_{u_1}}\right] \end{aligned}$$

$$
\begin{aligned}
&= \mathbf{1}_{k_1+k_2\geq N_{u_1}}(1-r^2)\sum_{v_1+\cdots+v_{n-1}=k_1-(u_1+\cdots+m_{n-1})}\prod_{j=2}^{n-1}\frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\,\eta_j^{v_j}\ \times\\
&\quad\times\frac{(k_1+k_2-N_{u_1})!}{v_1!\,(k_1+k_2-N_{u_1}-v_1)!}\left(r^2z_1\right)^{v_1}\left((1-r^2)z_2\right)^{k_1+k_2-N_{u_1}-v_1}\\
&- \mathbf{1}_{k_1+k_2\geq N_{u_1}}\sum_{v\leq n-1,\alpha_v>\alpha_{q_l}}\frac{1-r^2}{(m_v-1)!}\ \times\\
&\quad\times\frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v}\left[\frac{t^{k_1-u_1-1}}{\prod_{j=2,j\neq v}^{n-1}(t-\eta_j)^{m_j}}\left((1-r^2)z_2+r^2z_1/t\right)^{k_1+k_2-N_{u_1}}\right].
\end{aligned}
$$

Similarly, for all $l=2,\ldots,\widetilde{n}$ and $r=\alpha_{q_l}-\varepsilon$, we have

$$
\frac{1}{(2\pi i)^2}\int_{r=\alpha_{q_l}-\varepsilon}\frac{(1-r^2)\,\zeta_2^{k_2-m_n}}{\zeta_2-(1-r^2)z_2}\,\frac{\zeta_1^{k_1-u_1}}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}\left(\zeta_1-\frac{r^2z_1\zeta_2}{\zeta_2-(1-r^2)z_2}\right)}\,\omega(\zeta)\quad=
$$

$$
\begin{aligned}
&= \mathbf{1}_{k_1+k_2\geq N_{u_1}}(1-r^2)\sum_{v_1+\cdots+v_{n-1}=k_1-(u_1+\cdots+m_{n-1})}\prod_{j=2}^{n-1}\frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\,\eta_j^{v_j}\ \times\\
&\quad\times\frac{(k_1+k_2-N_{u_1})!}{v_1!\,(k_1+k_2-N_{u_1}-v_1)!}\left(r^2z_1\right)^{v_1}\left((1-r^2)z_2\right)^{k_1+k_2-N_{u_1}-v_1}\\
&- \mathbf{1}_{k_1+k_2\geq N_{u_1}}\sum_{v\leq n-1,\alpha_v\geq\alpha_{q_l}}\frac{1-r^2}{(m_v-1)!}\ \times\\
&\quad\times\frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v}\left[\frac{t^{k_1-u_1-1}}{\prod_{j=2,j\neq v}^{n-1}(t-\eta_j)^{m_j}}\left((1-r^2)z_2+r^2z_1/t\right)^{k_1+k_2-N_{u_1}}\right].
\end{aligned}
$$

It follows that

$$
\lim_{\varepsilon\to0}\frac{1}{(2\pi i)^2}\sum_{l=2}^{\widetilde{n}-1}\left(\int_{r=\alpha_{q_l}+\varepsilon}-\int_{r=\alpha_{q_l}-\varepsilon}\right)\frac{\overline{\zeta}_2\,\zeta_2^{k_2-m_n}\zeta_1^{k_1-u_1-1}\,\omega(\zeta)}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}\left(1-<\overline{\zeta},z>\right)}\quad=
$$

$$
\begin{aligned}
&= \mathbf{1}_{k_1+k_2\geq N_{u_1}}\sum_{l=2}^{\widetilde{n}-1}\sum_{2\leq v\leq n-1,\alpha_v=\alpha_{q_l}}\frac{1-\alpha_{q_l}^2}{(m_v-1)!}\ \times\\
&\times\ \frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v}\left[\frac{t^{k_1-u_1-1}}{\prod_{j=2,j\neq v}^{n-1}(t-\eta_j)^{m_j}}\left((1-\alpha_{q_l}^2)z_2+\alpha_{q_l}^2z_1/t\right)^{k_1+k_2-N_{u_1}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbf{1}_{k_1+k_2\geq N_{u_1}}\sum_{v=2}^{n-1}\frac{1-\alpha_v^2}{(m_v-1)!}\ \times\\
&\quad\times\frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v}\left[\frac{t^{k_1-u_1-1}}{\prod_{j=2,j\neq v}^{n-1}(t-\eta_j)^{m_j}}\left((1-\alpha_v^2)z_2+\alpha_v^2z_1/t\right)^{k_1+k_2-N_{u_1}}\right].
\end{aligned}
\tag{5.12}
$$

On the other hand,

$$\lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2}\int_{r=1-\varepsilon}\frac{\overline{\zeta}_2\,\zeta_2^{k_2-m_n}\zeta_1^{k_1-u_1-1}\;\omega(\zeta)}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}\,\left(1-<\overline{\zeta},z>\right)} \;=\tag{5.13}$$

$$\begin{aligned} = \quad & \mathbf{1}_{k_1+k_2\geq N_{u_1}}(1-r^2)\sum_{v_1+\cdots+v_{n-1}=k_1-(u_1+\cdots+m_{n-1})}\prod_{j=2}^{n-1}\frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\,\eta_j^{v_j}\;\times \\ & \times\,\frac{(k_1+k_2-N_{u_1})!}{v_1!\,(k_1+k_2-N_{u_1}-v_1)!}\left(r^2z_1\right)^{v_1}\left((1-r^2)z_2\right)^{k_1+k_2-N_{u_1}-v_1} \\ \xrightarrow[r=1-\varepsilon\to 1]{} \quad & 0\,. \end{aligned}$$

Lastly, we have for $r=\varepsilon$

$$\frac{1}{(2\pi i)^2}\int_{r=\varepsilon}\frac{\overline{\zeta}_2\,\zeta_2^{k_2-m_n}\zeta_1^{k_1-u_1-1}\;\omega(\zeta)}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}\,\left(1-<\overline{\zeta},z>\right)} \;=$$

$$\begin{aligned} = \;& \mathbf{1}_{k_1+k_2\geq N_{u_1}}(1-r^2)\sum_{v_1+\cdots+v_{n-1}=k_1-(u_1+\cdots+m_{n-1})}\prod_{j=2}^{n-1}\frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\,\eta_j^{v_j}\;\times \\ & \times\,\frac{(k_1+k_2-N_{u_1})!}{v_1!\,(k_1+k_2-N_{u_1}-v_1)!}\left(r^2z_1\right)^{v_1}\left((1-r^2)z_2\right)^{k_1+k_2-N_{u_1}-v_1} \\ - \;& \mathbf{1}_{k_1+k_2\geq N_{u_1}}\sum_{v=2}^{n-1}\frac{1-r^2}{(m_v-1)!}\;\times \\ & \times\,\frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v}\left[\frac{t^{k_1-u_1-1}}{\prod_{j=2,j\neq v}^{n-1}(t-\eta_j)^{m_j}}\left((1-r^2)z_2+r^2z_1/t\right)^{k_1+k_2-N_{u_1}}\right] \end{aligned}$$

$$\begin{aligned} \xrightarrow[r=\varepsilon\to 0]{} \quad & \mathbf{1}_{k_1+k_2\geq N_{u_1}}\,z_2^{k_1+k_2-N_{u_1}}\sum_{v_2+\cdots+v_{n-1}=k_1-(u_1+\cdots+m_{n-1})}\prod_{j=2}^{n-1}\frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\,\eta_j^{v_j} \\ & -\mathbf{1}_{k_1+k_2\geq N_{u_1}}\,z_2^{k_1+k_2-N_{u_1}}\sum_{v=2}^{n-1}\frac{1}{(m_v-1)!}\frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v}\left[\frac{t^{k_1-u_1-1}}{\prod_{j=2,j\neq v}^{n-1}(t-\eta_j)^{m_j}}\right]. \end{aligned}$$

If $k_1\geq u_1+1$, we get by lemma 3

$$\begin{aligned} & \sum_{v_2+\cdots+v_{n-1}=k_1-u_1-(m_2+\cdots+m_{n-1})}\prod_{j=2}^{n-1}\frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\,\eta_j^{v_j} \\ & \quad+\sum_{v=2}^{n-1}\frac{1}{(m_v-1)!}\frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v}\left[\frac{t^{k_1-u_1}}{\prod_{j=2,j\neq v}^{n-1}(t-\eta_j)^{m_j}(-t)}\right] \;= \end{aligned}$$

$$\begin{aligned}
&= \left[Q\left(X^{k_1-u_1},\prod_{j=2}^{n-1}(X-\eta_j)^{m_j}\right) + \frac{R\left(X^{k_1-u_1},\prod_{j=2}^{n-1}(X-\eta_j)^{m_j}\right)}{\prod_{j=2}^{n-1}(X-\eta_j)^{m_j}}\right]|_{X=0}\\
&= \left[\frac{X^{k_1-u_1}}{\prod_{j=2}^{n-1}(X-\eta_j)^{m_j}}\right]|_{X=0}\\
&= 0\,.
\end{aligned}$$

Else $k_1 \le u_1$. Since there exists $m_p \ge 1,\ 2 \le p \le n-1$, it follows that $k_1 \le u_1 < u_1 + m_2 \cdots + m_{n-1}$ then

$$\sum_{v_2+\cdots+v_{n-1}=k_1-u_1-(m_2+\cdots+m_{n-1})}\prod_{j=2}^{n-1}\frac{(v_j+m_j-1)!}{v_j!\,(m_j-1)!}\,\eta_j^{v_j} \quad = \quad 0$$

and we get, for all $k_1,\ k_2 \ge 0$,

$$\begin{aligned}
(5.14)\qquad & -\,\mathbf{1}_{k_1+k_2\ge N_{u_1},\,k_1\le u_1}\ \times\\
&\times\ z_2^{k_1+k_2-N_{u_1}}\sum_{v=2}^{n-1}\frac{1}{(m_v-1)!}\frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v}\left[\frac{t^{k_1-u_1-1}}{\prod_{j=2,j\neq v}^{n-1}(t-\eta_j)^{m_j}}\right].
\end{aligned}$$

We finally get by (5.12), (5.13), and (5.14)

$$\lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2}\left[\int_{r=1-\varepsilon}-\sum_{l=2}^{\widetilde{n}-1}\left(\int_{r=\alpha_{q_l}+\varepsilon}-\int_{r=\alpha_{q_l}-\varepsilon}\right)-\int_{r=\varepsilon}\right]\frac{\overline{\zeta}_2\,\zeta_2^{k_2-m_n}\zeta_1^{k_1-u_1-1}\,\omega(\zeta)}{\prod_{j=2}^{n-1}(\zeta_1-\eta_j\zeta_2)^{m_j}\,(1-<\overline{\zeta},z>)} =$$

$$\begin{aligned}
&= \ 0\ -\ \mathbf{1}_{k_1+k_2\ge N_{u_1}}\sum_{v=2}^{n-1}\frac{1}{1+|\eta_v|^2}\,\frac{1}{(m_v-1)!}\ \times\\
&\quad\times\ \frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v}\left[\frac{t^{k_1-u_1-1}}{\prod_{j=2,j\neq v}^{n-1}(t-\eta_j)^{m_j}}\left(\frac{z_2+|\eta_v|^2z_1/t}{1+|\eta_v|^2}\right)^{k_1+k_2-N_{u_1}}\right]\\
&+\ \mathbf{1}_{k_1+k_2\ge N_{u_1},\,k_1\le u_1}\,z_2^{k_1+k_2-N_{u_1}}\sum_{v=2}^{n-1}\frac{1}{(m_v-1)!}\frac{\partial^{m_v-1}}{\partial t^{m_v-1}}|_{t=\eta_v}\left[\frac{t^{k_1-u_1-1}}{\prod_{j=2,j\neq v}^{n-1}(t-\eta_j)^{m_j}}\right]
\end{aligned}$$

and the proof is achieved.

$\surd$

6. Proof of theorem 1

Consider $f \in \mathcal{O}(\mathbb{B}_2)$ and $f(z) = \sum_{k_1,k_2\ge 0} a_{k_1,k_2} z_1^{k_1} z_2^{k_2}$ its Taylor expansion for all $z \in \mathbb{B}_2$.

First, we have the following preliminar result.

Lemma 11. *The Taylor expansion of f is absolutely convergent on any compact subset $K \subset \mathbb{B}_2$.*

Proof. $z \in \mathbb{B}_2$ being fixed, consider the bidisc $D_z := D_2(0, r_z)$ where $r_z = (r_{z,1}, r_{z,2})$ is such that $|z_1| < r_{z,1}$ (resp. $|z_2| < r_{z,2}$) and $r_{z,z}^2 + r_{z,2}^2 < 1$ (this is possible since $|z_1|^2 + |z_2|^2 < 1$). Then $z \in D_z \subset \overline{D_z} \subset \mathbb{B}_2$ and, for all k_1, $k_2 \geq 0$, the Cauchy formula on $\overline{D_z}$ gives

$$a_{k_1,k_2} = \frac{1}{k_1!\,k_2!}\frac{\partial^{k_1+k_2} f}{\partial\zeta_1^{k_1}\partial\zeta_2^{k_2}}(0,0) \quad = \quad \frac{1}{(2\pi i)^2}\int_{|\zeta_1|=r_{z,1},\,|\zeta_2|=r_{z,2}} \frac{f(\zeta_1,\zeta_2)\,\omega(\zeta)}{\zeta_1^{k_1+1}\zeta_2^{k_2+1}}$$

then

$$|a_{k_1,k_2}| \quad \leq \quad \frac{\sup_{\zeta\in\overline{D_z}}|f(\zeta)|}{r_{z,1}^{k_1}\,r_{z,2}^{k_2}}. \tag{6.1}$$

If we set $D'_z := D_2(0, r'_z)$ where $|z_1| < r'_{z,1} < r_{z,1}$ (resp. $|z_2| < r'_{z,2} < r_{z,2}$), it follows that, for all $z' \in \overline{D'_z}$,

$$\begin{aligned}\sum_{k_1,k_2\geq 0}\left|a_{k_1,k_2}{z'_1}^{k_1}{z'_2}^{k_2}\right| \quad &\leq \quad \sup_{\zeta\in\overline{D_z}}|f(\zeta)|\sum_{k_1\geq 0}\left(\frac{|z'_1|}{r_{z,1}}\right)^{k_1}\times\sum_{k_2\geq 0}\left(\frac{|z'_2|}{r_{z,2}}\right)^{k_2}\\ &\leq \quad \frac{\sup_{\zeta\in\overline{D_z}}|f(\zeta)|}{(1-r'_{z,1}/r_{z,1})(1-r'_{z,2}/r_{z,2})} \;<\; +\infty\end{aligned}$$

and the taylor series is absolutely convergent on the neighborhood $\overline{D'_z}$ of z.

Now if $K \subset \mathbb{B}_2$ a compact subset, it can be covered by a finite number of such D'_{z_j} in which the Taylor series is absolutely convergent.

$\surd$

Now we can give the proof of theorem 1.

Proof. First, notice that if $m_2 = \cdots = m_{n-1} = 0$, we have (since the Taylor expansion of f is absolutely convergent)

$$\begin{aligned}f(z) \quad &= \quad \sum_{k_1,k_2\geq 0} a_{k_1,k_2}z_1^{k_1}z_2^{k_2}\\ &= \quad \left[\sum_{k_1\leq m_1-1,k_2\geq 0} + \sum_{k_1\geq 0,k_2\leq m_n-1} - \sum_{k_1\leq m_1-1,k_2\leq m_n-1}\right] a_{k_1,k_2}z_1^{k_1}z_2^{k_2}\\ &\quad + \sum_{k_1\geq m_1,k_2\geq m_n} a_{k_1,k_2}z_1^{k_1}z_2^{k_2}\\ &= \quad \sum_{k_1\leq m_1-1}\frac{z_1^{k_1}}{k_1!}\left(\frac{\partial^{k_1}f}{\partial z_1^{k_1}}\right)(0,z_2) + \sum_{k_2\leq m_n-1}\frac{z_2^{k_2}}{k_2!}\left(\frac{\partial^{k_2}f}{\partial z_2^{k_2}}\right)(z_1,0)\\ &\quad - \sum_{k_1\leq m_1-1,k_2\leq m_n-1}\frac{z_1^{k_1}}{k_1!}\frac{z_2^{k_2}}{k_2!}\left(\frac{\partial^{k_1+k_2}f}{\partial z_1^{k_1}\partial z_2^{k_2}}\right)(0) + \sum_{k_1\geq m_1,k_2\geq m_n} a_{k_1,k_2}z_1^{k_1}z_2^{k_2}\end{aligned}$$

and theorem 1 is proved in this case with

$$\mathcal{G}\left(\eta_1^{m_1},\eta_2^0,\ldots,\eta_{n-1}^0,\eta_n^{m_n};f\right)(z) \;=\; \sum_{k_1\leq m_1-1}\frac{z_1^{k_1}}{k_1!}\left(\frac{\partial^{k_1}f}{\partial z_1^{k_1}}\right)(0,z_2) \tag{6.2}$$

$$+\sum_{k_2\le m_n-1}\frac{z_2^{k_2}}{k_2!}\left(\frac{\partial^{k_2}f}{\partial z_2^{k_2}}\right)(z_1,0)-\sum_{k_1\le m_1-1,k_2\le m_n-1}\frac{z_1^{k_1}}{k_1!}\frac{z_2^{k_2}}{k_2!}\left(\frac{\partial^{k_1+k_2}f}{\partial z_1^{k_1}\partial z_2^{k_2}}\right)(0)\,.$$

Now we can assume that there exists $m_p\ge 1,\ 2\le p\le n-1$. An application of proposition 2 to each monomial $z_1^{k_1}z_2^{k_2}$, k_1, $k_2\ge 0$ (that is holomorphic on $\overline{\mathbb{B}_2}$ even if f is not) gives

$$\begin{aligned}(6.3)\qquad z_1^{k_1}z_2^{k_2} \;&=\; \lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2}\int_{\partial\widetilde{\Sigma}_\varepsilon}\frac{\zeta_1^{k_1}\zeta_2^{k_2}\det\left(\overline{\zeta},P_n(\zeta,z)\right)}{g_n(\zeta)\left(1-<\overline{\zeta},z>\right)}\,\omega(\zeta)\\ &\quad-\lim_{\varepsilon\to 0}\frac{g_n(z)}{(2\pi i)^2}\int_{\widetilde{\Sigma}_\varepsilon}\frac{\zeta_1^{k_1}\zeta_2^{k_2}\;\omega'\left(\overline{\zeta}\right)\wedge\omega(\zeta)}{g_n(\zeta)\;\left(1-<\overline{\zeta},z>\right)^2}\,.\end{aligned}$$

By proposition 3 we have, for all $z\in U_\eta$ and all k_1, $k_2\ge 0$,

$$(6.4)\quad -\lim_{\varepsilon\to 0}\frac{g_n(z)}{(2\pi i)^2}\int_{\Sigma_\varepsilon}\frac{\zeta_1^{k_1}\zeta_2^{k_2}\;\omega'\left(\overline{\zeta}\right)\wedge\omega(\zeta)}{g_n(\zeta)\;\left(1-<\overline{\zeta},z>\right)^2}\quad=$$

$$\begin{aligned}&=\quad \mathbf{1}_{k_1+k_2\ge N,\,k_1\ge m_1,k_2\ge m_n}\,z_1^{k_1}\,z_2^{k_2}\\
&-\quad \mathbf{1}_{k_1+k_2\ge N}\sum_{p=2}^{n-1}z_1^{m_1}\prod_{j=2,j\ne p}^{n-1}(z_1-\eta_j z_2)^{m_j}\,z_2^{m_n}\sum_{s=0}^{m_p-1}z_2^{m_p-1-s}(z_1-\eta_p z_2)^s\\
&\qquad\times\frac{1}{s!}\frac{\partial^s}{\partial t^s}\Big|_{t=\eta_p}\left[\frac{t^{k_1}}{t^{m_1}\prod_{j=2,j\ne p}^{n-1}(t-\eta_j)^{m_j}}\left(\frac{z_2+|\eta_p|^2z_1/t}{1+|\eta_p|^2}\right)^{k_1+k_2-N+1}\right]\\
&+\quad \mathbf{1}_{k_1\le m_1-1,k_2\ge N-k_1}\sum_{p=2}^{n-1}z_1^{m_1}\prod_{j=2,j\ne p}^{n-1}(z_1-\eta_j z_2)^{m_j}\,z_2^{m_n}\sum_{s=0}^{m_p-1}z_2^{m_p-1-s}(z_1-\eta_p z_2)^s\\
&\qquad\times\frac{1}{s!}\frac{\partial^s}{\partial t^s}\Big|_{t=\eta_p}\left[\frac{t^{k_1}z_2^{k_1+k_2-N+1}}{t^{m_1}\prod_{j=2,j\ne p}^{n-1}(t-\eta_j)^{m_j}}\right]\\
&+\quad \mathbf{1}_{k_2\le m_n-1,k_1\ge N-k_2}\sum_{p=2}^{n-1}z_1^{m_1}\prod_{j=2,j\ne p}^{n-1}(z_1-\eta_j z_2)^{m_j}\,z_2^{m_n}\sum_{s=0}^{m_p-1}z_2^{m_p-1-s}(z_1-\eta_p z_2)^s\\
&\qquad\times\frac{1}{s!}\frac{\partial^s}{\partial t^s}\Big|_{t=\eta_p}\left[\frac{t^{N-1-k_2}z_1^{k_1+k_2-N+1}}{t^{m_1}\prod_{j=2,j\ne p}^{n-1}(t-\eta_j)^{m_j}}\right]\,.\end{aligned}$$

Similarly, we have by proposition 4, for all $z\in U_\eta$ and all k_1, $k_2\ge 0$,

$$(6.5)\quad \lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2}\int_{\partial\widetilde{\Sigma}_\varepsilon}\frac{\zeta_1^{k_1}\zeta_2^{k_2}\det\left(\overline{\zeta},P_n(\zeta,z)\right)}{g_n(\zeta)\left(1-<\overline{\zeta},z>\right)}\,\omega(\zeta)\quad=$$

$$= \sum_{u_1=0}^{m_1-1} z_1^{u_1} \prod_{j=2}^{n-1} (z_1 - \eta_j z_2)^{m_j} z_2^{m_n} \sum_{p=2}^{n-1} \frac{1}{(m_p-1)!} \frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p} \frac{1}{t^{u_1+1} \prod_{j=2, j\neq p}^{n-1} (t-\eta_j)^{m_j}} \times$$

$$\times \{ \mathbf{1}_{k_1+k_2 \geq N_{u_1}} \frac{1}{1+|\eta_p|^2} t^{k_1} \left(\frac{z_2 + |\eta_p|^2 z_1/t}{1+|\eta_p|^2} \right)^{k_1+k_2-N_{u_1}}$$

$$- \mathbf{1}_{k_1 \leq u_1,\, k_2 \geq N_{u_1}-k_1} t^{k_1} z_2^{k_1+k_2-N_{u_1}} \}$$

$$+ \sum_{p=2}^{n-1} \sum_{u_p=0}^{m_p-1} (z_1 - \eta_p z_2)^{u_p} \prod_{j=p+1}^{n-1} (z_1 - \eta_j z_2)^{m_j} z_2^{m_n} \times$$

$$\times \quad \{ \mathbf{1}_{k_1+k_2 \geq N_{u_p}} \{ \frac{1}{u_p!} \frac{\partial^{u_p}}{\partial t^{u_p}}|_{t=\eta_p} \frac{1+|\eta_p|^2 \eta_p/t}{1+|\eta_p|^2} \frac{t^{k_1} \left(\frac{z_2+|\eta_p|^2 z_1/t}{1+|\eta_p|^2} \right)^{k_1+k_2-N_{u_p}}}{\prod_{j=p+1}^{n-1} (t-\eta_j)^{m_j}}$$

$$+ \sum_{q=p+1}^{n-1} \frac{1}{(m_q-1)!} \frac{\partial^{m_q-1}}{\partial t^{m_q-1}}|_{t=\eta_q} \frac{1+|\eta_q|^2 \eta_p/t}{1+|\eta_q|^2} \frac{t^{k_1} \left(\frac{z_2+|\eta_q|^2 z_1/t}{1+|\eta_q|^2} \right)^{k_1+k_2-N_{u_p}}}{(t-\eta_p)^{u_p+1} \prod_{j=p+1, j\neq q}^{n-1} (t-\eta_j)^{m_j}} \}$$

$$- \mathbf{1}_{k_2 \leq m_n-1, k_1 \geq N_{u_p}-k_2} \, \eta_p z_1^{k_1+k_2-N_{u_p}} \times \{ \frac{1}{u_p!} \frac{\partial^{u_p}}{\partial t^{u_p}}|_{t=\eta_p} \frac{t^{N_{u_p}-1-k_2}}{\prod_{j=p+1}^{n-1} (t-\eta_j)^{m_j}}$$

$$+ \sum_{q=p+1}^{n-1} \frac{1}{(m_q-1)!} \frac{\partial^{m_q-1}}{\partial t^{m_q-1}}|_{t=\eta_q} \frac{t^{N_{u_p}-1-k_2}}{(t-\eta_p)^{u_p+1} \prod_{j=p+1, j\neq q}^{n-1} (t-\eta_j)^{m_j}} \}$$

$$+ \quad \mathbf{1}_{k_1 \geq 0, k_2 \leq m_n-1} z_1^{k_1} z_2^{k_2} .$$

It follows from (6.3), (6.4), (6.5) and by continuity with repect to z that we have, for all $z \in \mathbb{B}_2$,

$$f(z) = \sum_{k_1,k_2 \geq 0} a_{k_1,k_2} z_1^{k_1} z_2^{k_2} =$$

$$= \sum_{u_1=0}^{m_1-1} z_1^{u_1} \prod_{j=2}^{n-1} (z_1 - \eta_j z_2)^{m_j} z_2^{m_n} \sum_{p=2}^{n-1} \frac{1}{(m_p-1)!} \frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p} \frac{1}{t^{u_1+1} \prod_{j=2, j\neq p}^{n-1} (t-\eta_j)^{m_j}} \times$$

$$\times \{ \frac{1}{1+|\eta_p|^2} \sum_{k_1+k_2 \geq N_{u_1}} a_{k_1,k_2} t^{k_1} \left(\frac{z_2 + |\eta_p|^2 z_1/t}{1+|\eta_p|^2} \right)^{k_1+k_2-N_{u_1}}$$

$$- \sum_{k_1 \leq u_1,\, k_2 \geq N_{u_1}-k_1} a_{k_1,k_2} t^{k_1} z_2^{k_1+k_2-N_{u_1}} \}$$

$$+ \quad \sum_{p=2}^{n-1} \sum_{u_p=0}^{m_p-1} (z_1-\eta_p z_2)^{u_p} \prod_{j=p+1}^{n-1} (z_1-\eta_j z_2)^{m_j} z_2^{m_n} \quad \times$$

$$\times\{ \quad \frac{1}{u_p!}\frac{\partial^{u_p}}{\partial t^{u_p}}|_{t=\eta_p} \{ \frac{\frac{1+|\eta_p|^2\eta_p/t}{1+|\eta_p|^2}}{\prod_{j=p+1}^{n-1}(t-\eta_j)^{m_j}} \sum_{k_1+k_2\geq N_{u_p}} a_{k_1,k_2} t^{k_1} \left(\frac{z_2+|\eta_p|^2 z_1/t}{1+|\eta_p|^2}\right)^{k_1+k_2-N_{u_p}} \}$$

$$+\sum_{q=p+1}^{n-1} \frac{1}{(m_q-1)!}\frac{\partial^{m_q-1}}{\partial t^{m_q-1}}|_{t=\eta_q} \{ \frac{\frac{1+|\eta_q|^2\eta_p/t}{1+|\eta_q|^2}}{(t-\eta_p)^{u_p+1}\prod_{j=p+1,j\neq q}^{n-1}(t-\eta_j)^{m_j}} \quad \times$$

$$\times \sum_{k_1+k_2\geq N_{u_p}} a_{k_1,k_2} t^{k_1} \left(\frac{z_2+|\eta_q|^2 z_1/t}{1+|\eta_q|^2}\right)^{k_1+k_2-N_{u_p}} \}$$

$$- \quad \{ \frac{1}{u_p!}\frac{\partial^{u_p}}{\partial t^{u_p}}|_{t=\eta_p} \frac{\eta_p \sum_{k_2\leq m_n-1, k_1\geq N_{u_p}-k_2} a_{k_1,k_2} t^{N_{u_p}-1-k_2} z_1^{k_1+k_2-N_{u_p}}}{\prod_{j=p+1}^{n-1}(t-\eta_j)^{m_j}}$$

$$+\sum_{q=p+1}^{n-1} \frac{1}{(m_q-1)!}\frac{\partial^{m_q-1}}{\partial t^{m_q-1}}|_{t=\eta_q} \frac{1}{(t-\eta_p)^{u_p+1}\prod_{j=p+1,j\neq q}^{n-1}(t-\eta_j)^{m_j}} \quad \times$$

$$\times\ \eta_p \sum_{k_2\leq m_n-1, k_1\geq N_{u_p}-k_2} a_{k_1,k_2} t^{N_{u_p}-1-k_2} z_1^{k_1+k_2-N_{u_p}} \ \} \ \}$$

$$+ \quad \sum_{k_1\geq 0, u_n\leq m_n-1} a_{k_1,u_n} z_1^{k_1} z_2^{u_n} \ + \sum_{k_1+k_2\geq N,\, k_1\geq m_1, k_2\geq m_n} a_{k_1,k_2} z_1^{k_1} z_2^{k_2}$$

$$- \quad \sum_{p=2}^{n-1} z_1^{m_1} \prod_{j=2,j\neq p}^{n-1} (z_1-\eta_j z_2)^{m_j} z_2^{m_n} \sum_{s=0}^{m_p-1} z_2^{m_p-1-s}(z_1-\eta_p z_2)^s$$

$$\times \frac{1}{s!}\frac{\partial^s}{\partial t^s}|_{t=\eta_p} \left[\frac{1}{t^{m_1}\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}} \sum_{k_1+k_2\geq N} a_{k_1,k_2}\, t^{k_1} \left(\frac{z_2+|\eta_p|^2 z_1/t}{1+|\eta_p|^2}\right)^{k_1+k_2-N+1} \right]$$

$$+ \quad \sum_{p=2}^{n-1} z_1^{m_1} \prod_{j=2,j\neq p}^{n-1} (z_1-\eta_j z_2)^{m_j} z_2^{m_n} \sum_{s=0}^{m_p-1} z_2^{m_p-1-s}(z_1-\eta_p z_2)^s$$

$$\times \frac{1}{s!}\frac{\partial^s}{\partial t^s}|_{t=\eta_p} \left[\frac{1}{t^{m_1}\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}} \sum_{k_1\leq m_1-1, k_2\geq N-k_1} a_{k_1,k_2}\, t^{k_1} z_2^{k_1+k_2-N+1} \right]$$

$$+ \quad \sum_{p=2}^{n-1} z_1^{m_1} \prod_{j=2,j\neq p}^{n-1} (z_1-\eta_j z_2)^{m_j} z_2^{m_n} \sum_{s=0}^{m_p-1} z_2^{m_p-1-s}(z_1-\eta_p z_2)^s$$

$$\times \frac{1}{s!}\frac{\partial^s}{\partial t^s}|_{t=\eta_p} \left[\frac{1}{t^{m_1}\prod_{j=2,j\neq p}^{n-1}(t-\eta_j)^{m_j}} \sum_{k_2\leq m_n-1, k_1\geq N-k_2} a_{k_1,k_2}\, t^{N-1-k_2} z_1^{k_1+k_2-N+1} \right].$$

We have used the following lemma that allows us to switch the above series and derivative with respect to t.

Lemma 12. *Let $K \subset \mathbb{B}_2$ be a compact subset, $q \geq l \geq 0$ and $p = 2, \ldots, n-1$. Then for all $z \in K$ and all (t, w) in a neighborhood of (η_p, η_p), the following series*

$$\sum_{k_1+k_2 \geq q} a_{k_1,k_2} t^{k_1} \left(\frac{z_2 + |w|^2 z_1/t}{1+|w|^2}\right)^{k_1+k_2-q},$$

$$\sum_{k_1 \leq l, k_2 \geq q-k_1} a_{k_1,k_2} t^{k_1}\ z_2^{k_1+k_2-q}\,, \qquad \sum_{k_2 \leq l, k_1 \geq q-k_1} a_{k_1,k_2} t^{l-k_2}\ z_1^{k_1+k_2-q}$$

are absolutely convergent. In particular, all their derivatives with respect to t are absolutely convergent as series of holomorphic functions.

Proof. Consider the first series. One has, for all $z \in K$,

$$\left|\frac{z_2 + |w|^2 z_1/t}{1+|w|^2}\right| \leq \|z\| \frac{\sqrt{1+|w|^4/|t|^2}}{1+|w|^2} \leq (1-\varepsilon_K) \frac{\sqrt{1+|w|^4/|t|^2}}{1+|w|^2}.$$

Since $\|(t,1)\| = \sqrt{1+|t|^2}$, one can choose a neighborhood $W(\eta_p, \eta_p)$ such that, for all $(t, w) \in W(\eta_p, \eta_p)$,

$$\frac{1-\varepsilon_K}{\sqrt{1+|\eta_p|^2}} \leq \frac{\sqrt{1+|w|^4/|t|^2}}{1+|w|^2} \leq \frac{\sqrt{1+|w|^4/|t|^2}}{1+|w|^2}\sqrt{1+|t|^2} \leq 1+\varepsilon_K.$$

In particular,

$$\left(t\,\frac{(1-\varepsilon_K)\sqrt{1+|w|^4/|t|^2}}{1+|w|^2}, \frac{(1-\varepsilon_K)\sqrt{1+|w|^4/|t|^2}}{1+|w|^2}\right) \in \overline{\mathbb{B}_2}\left(0, (1-\varepsilon_K^2)\right).$$

It follows that

$$\sum_{k_1+k_2 \geq q} \sup_{z \in K, (t,w) \in W(\eta_p,\eta_p)} \left| a_{k_1,k_2}\, t^{k_1} \left(\frac{z_2 + |w|^2 z_1/t}{1+|w|^2}\right)^{k_1+k_2-q} \right| \leq$$

$$\leq \sum_{k_1+k_2 \geq q} \sup_{(t,w) \in W(\eta_p,\eta_p)} \left| a_{k_1,k_2}\, t^{k_1} \left((1-\varepsilon_K) \frac{\sqrt{1+|w|^4/|t|^2}}{1+|w|^2}\right)^{k_1+k_2-q} \right|$$

$$\leq \left(\frac{\sqrt{1+|\eta_p|^2}}{(1-\varepsilon_K)^2}\right)^q \sum_{k_1+k_2 \geq q} \sup_{(t,w) \in W(\eta_p,\eta_p)} \left| a_{k_1,k_2}\, t^{k_1} \left((1-\varepsilon_K) \frac{\sqrt{1+|w|^4/|t|^2}}{1+|w|^2}\right)^{k_1+k_2} \right|$$

$$\leq \left(\frac{\sqrt{1+|\eta_p|^2}}{(1-\varepsilon_K)^2}\right)^q \sum_{k_1+k_2 \geq q} \sup_{\zeta \in \overline{\mathbb{B}_2}(0, 1-\varepsilon_K^2)} \left| a_{k_1,k_2} \zeta_1^{k_1} \zeta_2^{k_2} \right| < +\infty$$

($\overline{\mathbb{B}_2}\left(0, 1-\varepsilon_K^2\right)$ can be covered by a finite number of polydiscs in which the series is absolutely convergent, see lemma 11).

The second series is a polynomial function with respect to t. On the other hand, notice that $|z_2| \leq \|z\| \leq 1-\varepsilon_K$ and $\overline{D_2}(0,(\varepsilon_K, 1-\varepsilon_K)) \subset \mathbb{B}_2$. It follows that, for all $k_1 = 0, \dots, l$,

$$\begin{aligned}
\sum_{k_2 \geq q-k_1} \sup_{z \in K} \left| a_{k_1,k_2} z_2^{k_1+k_2-q} \right| \quad &\leq \\
\leq \quad & \frac{1}{\varepsilon_K^{k_1}(1-\varepsilon_K)^{q-k_1}} \sum_{k_2 \geq q-k_1} \left| a_{k_1,k_2} \varepsilon_K^{k_1} (1-\varepsilon_K)^{k_2} \right| \\
\leq \quad & \frac{1}{\varepsilon_K^{k_1}(1-\varepsilon_K)^{q-k_1}} \sum_{u_1+u_2 \geq q} \sup_{\zeta \in \overline{D_2}(0,(\varepsilon_K, 1-\varepsilon_K))} |a_{u_1,u_2} \zeta_1^{u_1} \zeta_2^{u_2}| \\
< \quad & +\infty \,,
\end{aligned}$$

as well as $\sum_{k_1=0}^{l} \sum_{k_2 \geq q-k_1} \sup_{z \in K,\, t \in W(\eta_p)} \left| t^{k_1} a_{k_1,k_2} z_2^{k_1+k_2-q} \right|$.

The proof for the last series is similar.

$\surd$

It follows that

$$\begin{aligned}
f(z) \quad = \quad & \sum_{u_1=0}^{m_1-1} z_1^{u_1} z_2^{m_2+\cdots+m_n} \prod_{j=2}^{n-1} (z_1/z_2 - \eta_j)^{m_j} \sum_{p=2}^{n-1} \frac{1}{(m_p-1)!} \frac{\partial^{m_p-1}}{\partial t^{m_p-1}}|_{t=\eta_p} \frac{1}{\prod_{j=2, j\neq p}^{n-1} (t-\eta_j)^{m_j}} \times \\
\times \quad & \Big\{ \frac{1}{1+|\eta_p|^2} \sum_{k_1+k_2 \geq N_{u_1}} a_{k_1,k_2} t^{k_1-u_1-1} \left(\frac{z_2 + |\eta_p|^2 z_1/t}{1+|\eta_p|^2} \right)^{k_1+k_2-N_{u_1}} \\
& - \sum_{k_1 \leq u_1,\, k_2 \geq N_{u_1}-k_1} a_{k_1,k_2} t^{k_1-u_1-1} z_2^{k_1+k_2-N_{u_1}} \Big\} \\
+ \quad & \sum_{p=2}^{n-1} \sum_{u_p=0}^{m_p-1} z_2^{N_{u_p}} (z_1/z_2 - \eta_p)^{u_p+1} \prod_{j=p+1}^{n-1} (z_1/z_2 - \eta_j)^{m_j} \, \frac{1}{z_1/z_2 - \eta_p} \,\times \\
\times\Big\{ \quad & \frac{1}{u_p!} \frac{\partial^{u_p}}{\partial t^{u_p}}|_{t=\eta_p} \frac{\frac{1+|\eta_p|^2 \eta_p/t}{1+|\eta_p|^2}}{\prod_{j=p+1}^{n-1} (t-\eta_j)^{m_j}} \sum_{k_1+k_2 \geq N_{u_p}} a_{k_1,k_2} t^{k_1} \left(\frac{z_2 + |\eta_p|^2 z_1/t}{1+|\eta_p|^2} \right)^{k_1+k_2-N_{u_p}} \\
& + \sum_{q=p+1}^{n-1} \frac{1}{(m_q-1)!} \frac{\partial^{m_q-1}}{\partial t^{m_q-1}}|_{t=\eta_q} \frac{\frac{1+|\eta_q|^2 \eta_p/t}{1+|\eta_p|^2}}{(t-\eta_p)^{u_p+1} \prod_{j=p+1, j\neq q}^{n-1} (t-\eta_j)^{m_j}} \,\times \\
& \times \sum_{k_1+k_2 \geq N_{u_p}} a_{k_1,k_2} t^{k_1} \left(\frac{z_2 + |\eta_q|^2 z_1/t}{1+|\eta_q|^2} \right)^{k_1+k_2-N_{u_p}}
\end{aligned}$$

$$- \{ \ \frac{1}{u_p!}\frac{\partial^{u_p}}{\partial t^{u_p}}|_{t=\eta_p}\{\frac{\eta_p \sum_{k_2\le m_n-1, k_1\ge N_{u_p}-k_2} a_{k_1,k_2} t^{N_{u_p}-1-k_2} z_1^{k_1+k_2-N_{u_p}}}{\prod_{j=p+1}^{n-1}(t-\eta_j)^{m_j}}$$

$$+ \sum_{q=p+1}^{n-1} \frac{1}{(m_q-1)!}\frac{\partial^{m_q-1}}{\partial t^{m_q-1}}|_{t=\eta_q} \ \frac{\eta_p \sum_{k_2\le m_n-1, k_1\ge N_{u_p}-k_2} a_{k_1,k_2} t^{N_{u_p}-1-k_2} z_1^{k_1+k_2-N_{u_p}}}{(t-\eta_p)^{u_p+1}\prod_{j=p+1, j\neq q}^{n-1}(t-\eta_j)^{m_j}} \ \} \ \}$$

$$+ \quad \sum_{u_n=0}^{m_n-1} z_2^{u_n} \sum_{k_1\ge 0} a_{k_1,u_n} z_1^{k_1} \ + \sum_{k_1+k_2\ge N,\, k_1\ge m_1, k_2\ge m_n} a_{k_1,k_2} \, z_1^{k_1} \, z_2^{k_2}$$

$$- \quad \sum_{p=2}^{n-1} \prod_{j=2, j\neq p}^{n-1} (z_1/z_2-\eta_j)^{m_j} \sum_{s=0}^{m_p-1} (z_1/z_2-\eta_p)^s$$

$$\times \quad \frac{1}{s!}\frac{\partial^s}{\partial t^s}|_{t=\eta_p} \left[\frac{1}{\prod_{j=2, j\neq p}^{n-1}(t-\eta_j)^{m_j}} (z_1/z_2)^{m_1} z_2^{N-1} \sum_{k_1+k_2\ge N} a_{k_1,k_2} \, t^{k_1-m_1} \left(\frac{z_2+|\eta_p|^2 z_1/t}{1+|\eta_p|^2}\right)^{k_1+k_2-N+1} \right]$$

$$+ \quad \sum_{p=2}^{n-1} \prod_{j=2, j\neq p}^{n-1} (z_1/z_2-\eta_j)^{m_j} \sum_{s=0}^{m_p-1} (z_1/z_2-\eta_p)^s$$

$$\times \quad \frac{1}{s!}\frac{\partial^s}{\partial t^s}|_{t=\eta_p} \left[\frac{1}{\prod_{j=2, j\neq p}^{n-1}(t-\eta_j)^{m_j}} (z_1/z_2)^{m_1} \sum_{k_1\le m_1-1, k_2\ge N-k_1} a_{k_1,k_2} \, t^{k_1-m_1} z_2^{k_1+k_2} \right]$$

$$+ \quad (z_1/z_2)^{m_1} z_2^{N-1} \sum_{p=2}^{n-1} \prod_{j=2, j\neq p}^{n-1} (z_1/z_2-\eta_j)^{m_j} \sum_{s=0}^{m_p-1} (z_1/z_2-\eta_p)^s$$

$$\times \quad \frac{1}{s!}\frac{\partial^s}{\partial t^s}|_{t=\eta_p} \left[\frac{1}{\prod_{j=2, j\neq p}^{n-1}(t-\eta_j)^{m_j}} \sum_{k_2\le m_n-1, k_1\ge N-k_2} a_{k_1,k_2} \, t^{N-m_1-1-k_2} z_1^{k_1+k_2-N+1} \right]$$

$$= \quad \sum_{u_1=0}^{m_1-1} (z_1/z_2)^{u_1} z_2^{N_{u_1}} \ \times$$

$$\times \{ \quad \mathcal{L}\left(\eta_2^{m_2},\ldots,\eta_{n-1}^{m_{n-1}}; \sum_{k_1+k_2\ge N_{u_1}} \frac{a_{k_1,k_2} t^{k_1-u_1-1}}{1+|w|^2} \left(\frac{z_2+|w|^2 z_1/t}{1+|w|^2}\right)^{k_1+k_2-N_{u_1}} \right) (z_1/z_2)$$

$$- \mathcal{L}\left(\eta_2^{m_2},\ldots,\eta_{n-1}^{m_{n-1}}; \sum_{k_1\le u_1,\, k_2\ge N_{u_1}-k_1} a_{k_1,k_2} t^{k_1-u_1-1} z_2^{k_1+k_2-N_{u_1}} \right) (z_1/z_2) \ \}$$

$$+ \sum_{p=2}^{n-1} \sum_{u_p=0}^{m_p-1} z_2^{N_{u_p}} \ \times$$

$$\times \{ \mathcal{L}\left(\eta_p^{u_p+1},\ldots,\eta_{n-1}^{m_{n-1}};\frac{1+|w|^2\eta_p/t}{1+|w|^2}\sum_{k_1+k_2\geq N_{u_p}}\frac{a_{k_1,k_2}t^{k_1}}{z_1/z_2-\eta_p}\left(\frac{z_2+|w|^2z_1/t}{1+|w|^2}\right)^{k_1+k_2-N_{u_p}}\right)(z_1/z_2)$$

$$- \quad \mathcal{L}\left(\eta_p^{u_p+1},\ldots,\eta_{n-1}^{m_{n-1}};\sum_{k_2\leq m_n-1,k_1\geq N_{u_p}-k_2}\frac{\eta_p a_{k_1,k_2}t^{N_{u_p}-1-k_2}z_1^{k_1+k_2-N_{u_p}}}{z_1/z_2-\eta_p}\right)(z_1/z_2)\ \}$$

$$+ \quad \mathcal{L}\left(0^{m_n};\frac{f(z_1,t)}{z_2-t}\right)(z_2) \ + \sum_{k_1+k_2\geq N,\, k_1\geq m_1,k_2\geq m_n} a_{k_1,k_2}\, z_1^{k_1}\, z_2^{k_2}$$

$$- \quad \mathcal{L}\left(\eta_2^{m_2},\ldots,\eta_{n-1}^{m_{n-1}};\frac{z_2^{N-1}(z_1/z_2)^{m_1}}{z_1/z_2-t}\sum_{k_1+k_2\geq N}a_{k_1,k_2}\,t^{k_1-m_1}\left(\frac{z_2+|w|^2z_1/t}{1+|w|^2}\right)^{k_1+k_2-N+1}\right)(z_1/z_2)$$

$$+ \quad \mathcal{L}\left(\eta_2^{m_2},\ldots,\eta_{n-1}^{m_{n-1}};\frac{(z_1/z_2)^{m_1}}{z_1/z_2-t}\sum_{k_1\leq m_1-1,k_2\geq N-k_1}a_{k_1,k_2}\,t^{k_1-m_1}z_2^{k_1+k_2}\right)(z_1/z_2)$$

$$+ \quad \mathcal{L}\left(\eta_2^{m_2},\ldots,\eta_{n-1}^{m_{n-1}};\frac{(z_1/z_2)^{m_1}z_2^{N-1}}{z_1/z_2-t}\sum_{k_2\leq m_n-1,k_1\geq N-k_2}a_{k_1,k_2}\,t^{N-m_1-1-k_2}z_1^{k_1+k_2-N+1}\right)(z_1/z_2)\,.$$

The proof of the theorem will be complete with the following result.

$\surd$

Lemma 13. *Assume that, for all $p=1,\ldots,n-1$ (resp. $p=n$) and $u_p=0,\ldots,m_p-1$ (resp. $u_n=0,\ldots,m_n-1$), we know $\left(\frac{\partial^{u_p}f}{\partial z_1^{u_p}}\right)|_{\{z_1=\eta_p z_2\}}$ (resp. $\left(\frac{\partial^{u_n}f}{\partial z_2^{u_n}}\right)|_{\{z_2=0\}}$).*

Then, for all $z\in\mathbb{B}_2$,

$$f(z) \quad - \sum_{k_1+k_2\geq N,\, k_1\geq m_1,\, k_2\geq m_n} a_{k_1,\,k_2}\, z_1^{k_1}\, z_2^{k_2} \tag{6.6}$$

can be known as an explicit formula constructed from this data.

Proof. First, we have seen that

$$\mathcal{L}\left(0^{m_n};\frac{f(z_1,t)}{z_2-t}\right)(z_2) \quad = \quad \sum_{u_n=0}^{m_n-1}\frac{z_2^{u_n}}{u_n!}\left(\frac{\partial^{u_n}f}{\partial z_2^{u_n}}\right)(z_1,0)\,.$$

Next, we have

$$f(tv,v) \ = \sum_{k_1,k_2\geq 0}a_{k_1,k_2}t^{k_1}v^{k_1+k_2} \ = \sum_{l\geq 0}v^l\sum_{k_1+k_2=l}a_{k_1,k_2}t^{k_1}$$

that is also as an analytic function with respect to v

$$f(tv,v) \quad = \quad \sum_{l\geq 0} \frac{v^l}{l!}\frac{\partial^l}{\partial v^l}|_{v=0}\left[f(tv,v)\right] \ .$$

By the uniqueness of the coefficients we get, for all $v \geq 0$,

$$\sum_{k_1+k_2=l} a_{k_1,k_2}t^{k_1} \quad = \quad \frac{1}{l!}\frac{\partial^l}{\partial v^l}|_{v=0}\left[f(tv,v)\right] \ ,$$

then

$$\sum_{k_1+k_2\geq N_{u_p}} a_{k_1,k_2}\, t^{k_1}\left(\frac{z_2+|w|^2 z_1/t}{1+|w|^2}\right)^{k_1+k_2-N_{u_p}} \quad =$$

$$= \quad \sum_{l\geq N_{u_p}} \left(\frac{z_2+|w|^2 z_1/t}{1+|w|^2}\right)^{l-N_{u_p}} \sum_{k_1+k_2=l} a_{k_1,k_2}\, t^{k_1}$$

$$= \quad \sum_{l\geq N_{u_p}} \left(\frac{z_2+|w|^2 z_1/t}{1+|w|^2}\right)^{l-N_{u_p}} \frac{1}{l!}\frac{\partial^l}{\partial v^l}|_{v=0}\left[f(tv,v)\right] \ .$$

Now notice that, for all $q=2,\dots,n-1$, for all $s=0,\dots,m_q-1$ and all $l\geq 0$, the following derivative is known:

$$\frac{1}{s!}\frac{\partial^s}{\partial t^s}|_{t=\eta_q}\left[\frac{1}{l!}\frac{\partial^l}{\partial v^l}|_{v=0}\left(f(tv,v)\right)\right] \quad = \quad \frac{1}{l!}\frac{\partial^l}{\partial v^l}|_{v=0}\left[\frac{v^s}{s!}\left(\frac{\partial^s f}{\partial z_1^s}\right)(\eta_q v,v)\right] \ .$$

Indeed, for all v close to 0, $(\eta_q v,v)\in\{z\in\mathbb{B}_2,\ z_1-\eta_q z_2=0\}$, then we know all the $\left(\frac{\partial^s f}{\partial z_1^s}\right)(\eta_q v,v)$, $q=2,\dots,n-1$, $s=0,\dots,m_q-1$. It follows by lemma 12 that, for all $z\in\mathbb{B}_2$, we know

$$\frac{1}{s!}\frac{\partial^s}{\partial t^s}|_{t=\eta_q}\left[\sum_{k_1+k_2\geq N_{u_p}} a_{k_1,k_2}\, t^{k_1}\left(\frac{z_2+|w|^2 z_1/t}{1+|w|^2}\right)^{k_1+k_2-N_{u_p}}\right] \ ,$$

as well as, for all $u_1=0,\dots,m_1-1$,

$$\mathcal{L}\left(\eta_2^{m_2},\dots,\eta_{n-1}^{m_{n-1}};\sum_{k_1+k_2\geq N_{u_1}}\frac{a_{k_1,k_2}t^{k_1-u_1-1}}{1+|w|^2}\left(\frac{z_2+|w|^2 z_1/t}{1+|w|^2}\right)^{k_1+k_2-N_{u_1}}\right)(z_1/z_2)\,,$$

for all $p=2,\dots,n-1$ and $u_p=0,\dots,m_p-1$,

$$\mathcal{L}\left(\eta_p^{u_p+1},\dots,\eta_{n-1}^{m_{n-1}};\frac{1+|w|^2\eta_p/t}{1+|w|^2}\sum_{k_1+k_2\geq N_{u_p}} a_{k_1,k_2}t^{k_1}\left(\frac{z_2+|w|^2 z_1/t}{1+|w|^2}\right)^{k_1+k_2-N_{u_p}}\right)(z_1/z_2)$$

and

$$\mathcal{L}\left(\eta_2^{m_2},\dots,\eta_{n-1}^{m_{n-1}};\sum_{k_1+k_2\geq N}\frac{a_{k_1,k_2}\,t^{k_1-m_1}}{z_1/z_2-t}\left(\frac{z_2+|w|^2 z_1/t}{1+|w|^2}\right)^{k_1+k_2-N+1}\right)(z_1/z_2)\,.$$

Similarly, for all $u_1 = 0, \dots, m_1 - 1$, we know $\left(\frac{\partial^{u_1} f}{\partial z_1^{u_1}}\right)(0, z_2)$ then for all $l \geq 0$ we know

$$\frac{1}{l!}\frac{\partial^l}{\partial z_2^l}|_{z_2=0}\left[\frac{1}{u_1!}\left(\frac{\partial^{u_1} f}{\partial z_1^{u_1}}\right)(0, z_2)\right] \;=\; \frac{1}{u_1!\,l!}\left(\frac{\partial^{u_1+l} f}{\partial z_1^{u_1}\partial z_2^l}\right)(0) \;=\; a_{u_1,l}\,.$$

It follows that we know, for all $u_1 = 0, \dots, m_1 - 1$,

$$\mathcal{L}\left(\eta_2^{m_2}, \dots, \eta_{n-1}^{m_{n-1}}; \sum_{k_1 \leq u_1,\, k_2 \geq N_{u_1} - k_1} a_{k_1,k_2} t^{k_1 - u_1 - 1} z_2^{k_1 + k_2 - N_{u_1}}\right)(z_1/z_2)\,,$$

as well as

$$\mathcal{L}\left(\eta_2^{m_2}, \dots, \eta_{n-1}^{m_{n-1}}; \frac{1}{z_1/z_2 - t}\sum_{k_1 \leq m_1 - 1, k_2 \geq N - k_1} a_{k_1,k_2}\, t^{k_1 - m_1} z_2^{k_1+k_2}\right)(z_1/z_2)\,.$$

Lastly, for all $u_n = 0, \dots, m_n - 1$, we know $\left(\frac{\partial^{u_n} f}{\partial z_2^{u_n}}\right)(z_1, 0)$, as well as for all $l \geq 0$

$$\frac{1}{l!}\frac{\partial^l}{\partial z_1^l}|_{z_1=0}\left[\frac{1}{u_n!}\left(\frac{\partial^{u_n} f}{\partial z_2^{u_n}}\right)(z_1, 0)\right] \;=\; a_{l,u_n}\,.$$

It follows that we know, for all $p = 2, \dots, n-1$ and $u_p = 0, \dots, m_p - 1$,

$$\mathcal{L}\left(\eta_p^{u_p+1}, \dots, \eta_{n-1}^{m_{n-1}}; \frac{1 + |w|^2\eta_p/t}{1 + |w|^2}\sum_{k_1 + k_2 \geq N_{u_p}} a_{k_1,k_2} t^{k_1}\left(\frac{z_2 + |w|^2 z_1/t}{1 + |w|^2}\right)^{k_1 + k_2 - N_{u_p}}\right)(z_1/z_2)\,,$$

as well as

$$\mathcal{L}\left(\eta_2^{m_2}, \dots, \eta_{n-1}^{m_{n-1}}; \frac{1}{z_1/z_2 - t}\sum_{k_2 \leq m_n - 1, k_1 \geq N - k_2} a_{k_1,k_2}\, t^{N - m_1 - 1 - k_2} z_1^{k_1 + k_2 - N + 1}\right)(z_1/z_2)$$

and the lemma is proved.

$\surd$

One can specify $\mathcal{G}\,(\eta_1^{m_1}, \dots, \eta_n^{m_n}; f)$ in the special case with $m_2 = \cdots = m_{n-1} = 1$ and $m_1 = m_n = 0$. Then $N = n - 2$, for all $p = 2, \dots, n-1$, $N_{u_p} = n - p - 1 = N - p + 1$ and

(6.7) $\quad \mathcal{G}(\eta_2, \dots, \eta_{n-1}; f)(z) \quad =$

$$\begin{aligned} &= \sum_{p=2}^{n-1}\prod_{j=p+1}^{n-1}(z_1-\eta_j z_2)\sum_{q=p}^{n-1}\frac{1+\eta_p\overline{\eta_q}}{1+|\eta_q|^2}\frac{1}{\prod_{j=p,j\neq q}^{n-1}(\eta_q-\eta_j)} \times \\ &\quad \times \sum_{k_1+k_2\geq N-p+1} a_{k_1,k_2}\eta_q^{k_1}\left(\frac{z_2+\overline{\eta_q}z_1}{1+|\eta_q|^2}\right)^{k_1+k_2-(N-p+1)} \\ &- \sum_{p=2}^{n-1}\prod_{j=2,j\neq p}^{n-1}\frac{z_1-\eta_j z_2}{\eta_p-\eta_j}\sum_{k_1+k_2\geq N} a_{k_1,k_2}\eta_p^{k_1}\left(\frac{z_2+\overline{\eta_p}z_1}{1+|\eta_p|^2}\right)^{k_1+k_2-N+1} \end{aligned}$$

$$\begin{aligned} &= \sum_{p=2}^{n-1}\prod_{j=p+1}^{n-1}(z_1-\eta_j z_2)\sum_{q=p}^{n-1}\frac{1+\eta_p\overline{\eta_q}}{1+|\eta_q|^2}\frac{1}{\prod_{j=p,j\neq q}^{n-1}(\eta_q-\eta_j)} \times \\ &\quad \times \sum_{l\geq N-p+1}\left(\frac{z_2+\overline{\eta_q}z_1}{1+|\eta_q|^2}\right)^{l-(N-p+1)}\frac{1}{l!}\frac{\partial^l}{\partial v^l}|_{v=0}[f(\eta_q v,v)] \\ &- \sum_{p=2}^{n-1}\prod_{j=2,j\neq p}^{n-1}\frac{z_1-\eta_j z_2}{\eta_p-\eta_j}\sum_{l\geq N}\left(\frac{z_2+\overline{\eta_p}z_1}{1+|\eta_p|^2}\right)^{l-N+1}\frac{1}{l!}\frac{\partial^l}{\partial v^l}|_{v=0}[f(\eta_p v,v)]\,. \end{aligned}$$

We finish with the last result where we specify the precision for the approximation of f by $\mathcal{G}\left(\eta_1^{m_1},\ldots,\eta_n^{m_n};f\right)$ when $N\to+\infty$.

Corollary 1. *For all compact subset $K\subset\mathbb{B}_2$, we have*

$$\sup_{z\in K}\left|f(z)-\mathcal{G}\left(\eta_1^{m_1},\ldots,\eta_n^{m_n};f\right)(z)\right| \quad\leq\quad C(K,f)\left(1-\varepsilon_K\right)^N\ , \tag{6.8}$$

where $C(K,f)=C_K\,\sup_{\zeta\in K'}|f(\zeta)|$, $K'\supset K$ and ε_K, C_K depend on K.

In particular, if $\mathcal{F}\subset\mathcal{O}(\mathbb{B}_2)$ is a compact subset (i.e. a subset of holomorphic functions that is uniformly bounded on all compact subset of $\mathbb{B}_2$), then

$$\sup_{f\in\mathcal{F}}\sup_{z\in K}\left|f(z)-\mathcal{G}\left(\eta_1^{m_1},\ldots,\eta_n^{m_n};f\right)(z)\right| \quad\leq\quad C(K,\mathcal{F})(1-\varepsilon_K)^N\,. \tag{6.9}$$

Proof. It follows from theorem 1 that

$$f(z)-\mathcal{G}\left(\eta_1^{m_1},\ldots,\eta_n^{m_n};f\right)(z) \quad=\quad \sum_{k_1+k_2\geq N,k_1\geq m_1,k_2\geq m_n} a_{k_1,k_2}z_1^{k_1}z_2^{k_2}\,.$$

On the other hand, we know by lemma 11 that the Taylor expansion of f is absolutely convergent in K. More precisely, K being covered by a finite number of bidiscs $D(0,r_j)$, $j=1,\ldots,J$, one can choose $D(0,r_j')\supset\overline{D}(0,r_j)$, $j=1,\ldots,J$,

such that (see (6.1) in the proof of lemma 11)

$$\begin{aligned}\sup_{z\in K}\left|a_{k_1,k_2}z_1^{k_1}z_2^{k_2}\right| &\leq \max_{1\leq j\leq J}\sup_{z\in D(0,r_j)}\left|a_{k_1,k_2}z_1^{k_1}z_2^{k_2}\right| \\ &\leq \max_{1\leq j\leq J}\left[\sup_{\zeta\in D(0,r'_j)}|f(\zeta)|\left(\frac{r_{j,1}}{r'_{j,1}}\right)^{k_1}\left(\frac{r_{j,2}}{r'_{j,2}}\right)^{k_2}\right] \\ &\leq \sup_{\zeta\in K'}|f(\zeta)|\,(1-\varepsilon_K)^{k_1+k_2}\end{aligned}$$

(where $K'=\bigcup_{j=1}^{J}\overline{D}(0,r'_j)$). Then

$$\begin{aligned}\sup_{z\in K}\left|\sum_{k_1+k_2\geq N,k_1\geq m_1,k_2\geq m_n}a_{k_1,k_2}z_1^{k_1}z_2^{k_2}\right| &\leq \sup_{\zeta\in K'}|f(\zeta)|\sum_{k_1+k_2\geq N}(1-\varepsilon_K)^{k_1+k_2} \\ &= \sup_{\zeta\in K'}|f(\zeta)|\sum_{q\geq 0}(q+N+1)(1-\varepsilon_K)^{q+N} \\ &\leq C_K\sup_{\zeta\in K'}|f(\zeta)|\left(\sqrt{1-\varepsilon_K}\right)^{N},\end{aligned}$$

and the corollary is proved by choosing $\varepsilon'_K<\varepsilon_K$.

$\surd$

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